Functions
Definition: Let $A$ and $B$ be sets. A function from $A$ to $B$, denoted $f : A \to B$, is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ to denote the assignment of $b$ to $a$ by the function $f$, where $a \in A$ and $b \in B$. 
Important Sets Associated with Functions

Definitions: Let $f$ be a function from $A$ to $B$. Let $f(a) = b$ for $a \in A$ and $b \in B$.

- $A$ is the *domain* of $f$ and $B$ is the *codomain* of $f$. (Why not "range"?)
- $b$ is the *image* of $a$, and $a$ is a *pre-image* of $b$. (Why not "the"?)
- The *range* of $f$ is the set of all images of elements of $A$. That is, the range of $f = \{ b \in B \mid \forall a \in A, \ f(a) = b \}$.
- We also say $f$ maps $A$ to $B$.

Definition: Let $f$ be a function from $A$ to $B$. Let $S$ be a subset of $A$. The *image* of $S$ is a subset of $B$ that consists of the images of the elements of $S$. We denote the image of $S$ by $f(S)$, so that $f(S) = \{ f(s) \mid s \in S \}$.

Note that the first definition of “image” defines the image of a single element, and the second definition defines the image of a set.
Important Properties of Functions

**Definition:** A function \( f \) is said to be **one-to-one**, or **injective**, if and only if \( f(x) = f(y) \) implies \( x = y \) for all \( x, y \) in the domain of \( f \). A function is said to be an **injection** if it is one-to-one.

Alternate definition: A function is one-to-one if and only if \( f(x) \neq f(y) \) whenever \( x \neq y \). (Note that this is the contrapositive of the other definition.)

**Definition:** A function \( f \) is said to be **onto**, or **surjective**, if and only if for every \( b \in B \) there is an element \( a \in A \) such that \( f(a) = b \). It is also said that set \( B \) is **covered** by \( f \).

**Definition:** A function if said to be **bijective** (or called a **bijection**) if it is both one-to-one and onto.
Theorem: Let $f$ be a function $f: A \to A$ from a finite set $A$ to itself. Then $f$ is one-to-one if and only if $f$ is onto.
**Functions on Real Numbers**

**Definition:** Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$ (real numbers). Then $f_1 + f_2$ and $f_1 f_2$ are also functions from $A$ to $\mathbb{R}$, defined by:

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 f_2)(x) = f_1(x) * f_2(x)$

**Definition:** A function $f$ whose domain and codomain are subsets of $\mathbb{R}$ is *strictly increasing* if, for $a$ and $b$ in the domain of $f$, $f(a) > f(b)$ whenever $a > b$. Similarly, $f$ is *strictly decreasing* if $f(a) < f(b)$ whenever $a > b$.

**Definition:** Let $A$ be a set. The *identity function* on $A$ is the function $i_A : A \rightarrow A$ where $i_A(x) = x$. 
Definition: Let $f$ be a bijection from set $A$ to set $B$. The inverse function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$. Hence, $f^{-1}(b) = a$, when $f(a) = b$. If the inverse function of $f$ exists, $f$ is called invertible.
Composition of Functions

**Definition:** Let $f$ be a function from set $A$ to set $B$, and let $g$ be a function from set $B$ to set $C$. The *composition* of the functions $g$ and $f$, denoted for all $a \in A$ by $(g \circ f)$, is defined by $(g \circ f)(a) = g(f(a))$. 