Formal and Informal Proofs
Logical Arguments

Definitions:

- An *argument* in propositional logic is a sequence of propositions.
- All but the final proposition in the argument are called *premises*.
- The final proposition is called the *conclusion*.
- An argument is *valid* if the truth of all of its premises implies that the conclusion is true.

Note that if an argument with premises $p_1, p_2, \ldots, p_n$ and conclusion $q$ is valid, then

$$(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$$

is a tautology. The above form is called the *tautological form* of the argument.
Formal Proofs

A formal proof of a statement:

- provides an argument supporting the validity of the statement
- shows that the conclusion follows from the premises
- uses Premises, Axioms, and results of other theorems
- show that each step of the proof follows logically from the Premises, Axioms, and other theorems.

In order to show that each step "follows logically", we use Rules of Inference.

Rules of Inference:

- represent logically valid argument forms or inference patterns
- allow us to infer new True statements from existing True statements
Informal Proofs

In *informal proofs*:

- The steps of the proofs are not expressed in any formal language such as propositional logic.
- Steps are argued less formally using English, mathematical formulas, etc.
- Logic and Rules of Inference are used to help us decide whether or not each step of the argument is sound.
Methods of Proving Theorems

Direct proof:

- $p \rightarrow q$ is proved by showing that if $p$ is True, $q$ follows.

Indirect proof:

- to prove $p \rightarrow q$, prove the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds, $\neg p$ follows.

Proof by contradiction:

- show that $(p \land \neg q)$ contradicts the assumptions

Proof by cases

- break the proof into cases and prove each one individually

Proof of equivalences:

- $(p \leftrightarrow q)$ is proved by proving $(p \rightarrow q) \land (q \rightarrow p)$. 
Proofs with Quantifiers

Existential Quantifiers:
1. Find an example (by making educated guesses) that shows that the statement holds.
2. Prove by contradiction: negate the existentially quantified statement and show that it implies a contradiction.
3. To disprove: choose an arbitrary $x$ and show that no matter what $x$ is, the property is False.

Universal Quantifiers:
1. Choose an arbitrary $x$ and show that no matter what $x$ is, the property holds true.
2. Proof by Cases: split the statement into cases and prove each of them.
3. Find a counterexample.