Graphs
Definitions

**Definition**: A graph \( G = (V,E) \) consists of a nonempty set \( V \) of *vertices* (or nodes) and a set \( E \) of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

**Definition**: A *directed graph* or *digraph* \( G = (V,E) \) consists of a nonempty set \( V \) of *vertices* (or nodes) together with a set \( E \) of *directed edges* (or *arcs*). Each edge is associated with an ordered pair of vertices.

- The directed edge associated with the ordered pair \( (u,v) \) is said to *start at* \( u \) and *end at* \( v \). The vertex \( u \) is called the *initial vertex* of the edge \( (a,b) \) and vertex \( v \) is the *terminal vertex* of this edge.

**Definition**: Graphs where the endpoints of an edge are not ordered are said to be *undirected graphs*. 
Types of graphs

**Definition:** In a *simple graph*, each edge connects two different vertices, and no two edges connect the same pair of vertices.

**Definition:** *Multigraphs* may have multiple edges connecting the same two vertices. When $m$ different edges connect the vertices $u$ and $v$, we say that $(u,v)$ is an edge of *multiplicity* $m$.

**Definition:** A *pseudograph* may include loops, as well as multiple edges connecting the same pair of vertices.
Definition: Two vertices in an undirected graph $G$ are called *adjacent* or *neighbors* in $G$ if there is an edge $e$ between $u$ and $v$. Such an edge $e$ is called *incident with* the vertices $u$ and $v$.

Definition: The set of all neighbors of a vertex $v$ of $G = (V,E)$, denoted by $N(v)$, is called the *neighborhood* of $v$. If $A \subseteq V$, we denote by $N(A)$ the set of all vertices in $G$ that are adjacent to at least one vertex in $A$.

Definition: The *degree* of a vertex in an undirected graph is the number of edges that are incident with it, except that a loop of a vertex contributes two to the degree of that vertex. The degree of the vertex $v$ is denoted by $\deg(v)$. The maximum degree of all of the vertices in a graph is designated by $\max-deg(G)$.
Handshaking Theorem

**Theorem** (the Handshaking Theorem): If $G = (V,E)$ is an undirected graph, then

\[ 2 |E| = \sum_{v \in V} \deg(v) \]
Definition: In a directed graph, the *in-degree* of a vertex \( v \), denoted \( \text{deg}^-(v) \), is the number of edges which terminate at \( v \). The *out-degree* of \( v \), denoted \( \text{deg}^+(v) \), is the number of edges with \( v \) as their initial vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.
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Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

**Theorem:** Let $G = (V,E)$ be a graph with directed edges. Then:

$$|E| = \sum_{v \in V} \text{deg}^-(v) = \sum_{v \in V} \text{deg}^+(v).$$
Definition: A *complete* graph on $n$ vertices, denoted by $K_n$, is the simple graph that contains exactly one edge between each pair of distinct vertices.
**Complete graphs**

**Definition:** A *complete* graph on $n$ vertices, denoted by $K_n$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

**Theorem:** The number of edges in $K_n$ is $n(n-1)/2$. 
Cycles

**Definition:** A cycle $C_n$ for $n \geq 3$ consists of $n$ vertices $v_1, v_2, v_3, \ldots, v_n$, and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$. 
**Definition:** An *n-dimensional hypercube*, or *n-cube*, $Q_n$, is a graph with $2^n$ vertices representing all bit strings of length $n$, where there is an edge between two vertices that differ in exactly one bit position.
Bipartite Graphs

**Definition**: A simple graph G is *bipartite* if V can be partitioned into two nonempty disjoint subsets V₁ and V₂ such that every edge connects a vertex in V₁ and a vertex in V₂. In other words, there are no edges which connect two vertices in V₁ or in V₂.

**Theorem**: A simple graph is bipartite if and only if it is possible to assign one of two colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.
Bipartite Graphs

**Definition:** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets $V_1$ of size $m$ and $V_2$ of size $n$ such that there is an edge from every vertex in $V_1$ to every vertex in $V_2$. 
Definition: A subgraph of a graph $G = (V,E)$ is a graph $(W,F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph $H$ of $G$ is a proper subgraph of $G$ if $H \neq G$.

Definition: Let $G = (V,E)$ be a simple graph. The subgraph induced by $W \subseteq V$ is the graph $(W,F)$, where the edge set $F$ contains an edge in $E$ if and only if both endpoints are in $W$. 
Union of Graphs

**Definition:** The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$. 
Definition: Suppose that \( G = (V,E) \) is a simple graph where \(|V| = n\). Arbitrarily list the vertices of \( G \) as \( v_1, v_2, \ldots, v_n \). The adjacency matrix \( A_G \) of \( G \), with respect to the listing of vertices, is the \( n \times n \) zero-one matrix with 1 as its \((i,j)\)th entry when \( v_i \) and \( v_j \) are adjacent, and 0 as its \((i,j)\)th entry when they are not adjacent.

In other words, the adjacency matrix \( A_G = [a_{ij}] = 1 \) if \((v_i,v_j)\) is an edge of \( G \), 0 otherwise.
Isomers

Butane ($C_4H_{10}$)

Isobutane ($C_4H_{10}$)
**Isomorphic Graphs**

**Definition:** The simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are *isomorphic* if there exists a one-to-one and onto function $f$ from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$.

Such a function $f$ is called an *isomorphism*.

Two simple graphs that are not isomorphic are called *nonisomorphic*.

In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.
**Definition:** Let $n$ be a nonnegative integer and $G$ an undirected graph. A path of length $n$ from $u$ to $v$ in $G$ is a sequence of $n$ edges $e_1, e_2, \ldots, e_n$ of $G$ for which there exists a sequence $x_0 = u, x_1, x_2, \ldots, x_n = v$ of vertices such that $e_i$ has, for $i = 1..n$, the endpoints $x_{i-1}$ and $x_i$. 

$n = 6$
Definition: A path is a *circuit* if it begins and ends in the same vertex; that is, if $u = v$, and has length greater than zero.

Definition: A path or circuit is said to *pass through* the vertices $x_1 \ldots x_{n-1}$, or *traverse* the edges $e_1 \ldots e_n$.

Definition: A path or circuit is *simple* if it does not contain the same edge more than once.
Paths in Directed Graphs

**Definition:** Let n be a nonnegative integer and G a directed graph. A *path of length n from u to v* in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that $e_1$ is associated with $(x_0, x_1)$, $e_2$ is associated with $(x_1, x_2)$, and so on, with $e_n$ associated with $(x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$.

**Definition:** A path of length greater than zero that begins and ends in the same vertex is called a *circuit* or *cycle*.

**Definition:** A path or circuit is called *simple* if it does not contain the same edge more than once.
Connectedness in Undirected Graphs

**Definition:** An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

An undirected graph that is not connected is called *disconnected*.

We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

**Theorem:** There is a simple path between every pair of distinct vertices of a connected undirected graph.
Definition: A connected component of a graph $G$ is a connected subgraph of $G$ that is not a proper subgraph of another connected subgraph of $G$.

- A graph $G$ that is not connected has two or more connected components that are disjoint and have $G$ as their union. In other words, the connected components make up a partition of $G$. 
For a graph $G$, if the removal of a vertex and all of its incident edges produces a subgraph consisting of more connected components, then the vertex is called a \textit{cut vertex}.

For a graph $G$, if the removal of an edge produces a subgraph consisting of more connected components, then the edge is called a \textit{cut edge}.

A graph may have zero, one, or many cut vertices and/or cut edges. A graph with no cut vertices is called a \textit{nonseparable} graph.
Vertex Cuts and Edge Cuts

**Definition:** For a connected graph $G = (V,E)$, $V' \subseteq V$ is a *vertex cut* if $G - V'$ is disconnected. In other words, if a set of vertices taken out together causes the disconnect.

**Definition:** For a connected graph $G = (V,E)$, $E' \subseteq E$ is an *edge cut* if $G - E'$ is disconnected. In other words, if a set of edges taken out together causes the disconnect.

**Theorem:** Every graph except a complete graph has a vertex cut.
Definition: The vertex connectivity of a noncomplete graph $G$, denoted by $\kappa(G)$, is the minimum number of vertices in a vertex cut of $G$.

The vertex connectivity of a complete graph $G$ is defined to be $n-1$, where $n = |V|$; in other words, $\kappa(K_n) = n-1$.

In general, the vertex connectivity $\kappa(G)$ of a graph is the minimum number of vertices that can be removed from $G$ to either disconnect $G$ or produce a graph with a single vertex.

Definition: The edge connectivity of a graph $G$, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of $G$. If $G$ consists of a single vertex, we define the edge connectivity of $G$ to be 0.
Strong and Weak Connectivity

**Definition:** A directed graph is *strongly connected* if, for every pair of vertices $a$ and $b$ in the graph, there is a path from $a$ to $b$ and from $b$ to $a$.

A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.
Theorem: Let $G$ be a graph with adjacency matrix $A$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length $r$ from $v_i$ to $v_j$, where $r$ is a positive integer, equals the $(i,j)$th entry of $A^r$. 

Number of Paths of Length $r$
The Bridges of Königsberg
The Bridges of Königsberg
Euler Circuits and Paths

**Definition:** In a graph $G$, an *Euler circuit* is a simple circuit containing every edge of $G$.

An *Euler path* in $G$ is a simple path containing every edge of $G$.

**Theorem:** A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

**Theorem:** A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.
Hamilton Circuits and Paths

**Definition:** In a graph $G$, a Hamilton circuit is a simple circuit that passes through every vertex exactly once.

A *Hamilton* path is a simple path that passes through every vertex exactly once.

Some observations about Hamilton circuits:

- A graph with a vertex of degree 1 cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex must be incident with two edges in the circuit.
- If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
Shortest-Path Problems: Airline Example (mileage)
Shortest-Path Problems: Airline Example (flight times)

Flight Times

San Francisco 1:15
Denver 2:20
Los Angeles 2:00
Chicago 2:55
Atlanta 3:50
New York 1:50
Boston 2:10
Miami 2:45

Flight times are indicated on the edges of the graph.
Shortest-Path Problems: Airline Example (fares)
**Definition**: A graph that has a number assigned to each edge is called a *weighted graph*.

The *length* of a path in a *weighted* graph is defined as the sum of the weights of the edges of the graph (instead of just the number of edges).
**Definition:** A *greedy algorithm* is any algorithm that attempts to solve an optimization problem by making the locally optimal choice at each step, with the hope of finding a global optimum.

In other words, it’s simply an algorithm that makes what seems to be the best choice at each step.

Greedy algorithms are usually fairly simple to implement, but they are NOT guaranteed to find the best answer.
Greedy Algorithms: Maze Example
Dijkstra's Algorithm

label the start vertex with $L(\text{start}) = 0$
label all other vertices with cost $L(\text{vertex}) = \infty$
mark all vertices "not visited"
set all vertices' path history to ()

while end vertex has not been marked "visited":
    find $u$, the vertex with the cheapest path
    mark $u$ visited
    for each $v$, a neighbor of $u$:
        if $\text{weight}(u,v) + L(u) < L(v)$:
            $L(v) = \text{weight}(u,v) + L(u)$
            set $v$'s history to $u$'s path history + $u$

Theorem: Dijkstra's Algorithm finds the length of a shorter path between two vertices in a connected simple undirected weighted graph.
The Traveling Salesman Problem

Diagram of cities: Grand Rapids, Saginaw, Kalamazoo, Toledo, Detroit

Distances:
- Grand Rapids to Kalamazoo: 56
- Grand Rapids to Saginaw: 113
- Kalamazoo to Saginaw: 167
- Kalamazoo to Toledo: 133
- Saginaw to Toledo: 137
- Saginaw to Detroit: 142
- Detroit to Toledo: 58
The Traveling Salesman Problem (cont.)

<table>
<thead>
<tr>
<th>Route</th>
<th>Total Distance (miles)</th>
</tr>
</thead>
</table>
Water, Gas, and Electricity
Water, Gas, and Electricity
Circuit Board
Planar Graphs

**Definition**: A graph is called *planar* if it can be drawn in the plane without any edges crossing, where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.

Such a drawing is called a *planar representation* of the graph.
Theorem (Euler’s Formula): Let G be a connected planar simple graph with $e$ edges and $v$ vertices. Let $r$ be the number of regions in a planar representation of G. Then

$$r = e - v + 2.$$
**Corollaries of Euler's Formula**

**Definition:** The *degree* of a region is the number of edges on the boundary of the region.

**Corollary 1:** If $G$ is a connected planar simple graph with $e$ edges and $v$ vertices, where $v \geq 3$, then $e \leq 3v - 6$.

**Corollary 2:** If $G$ is a connected planar simple graph, then $G$ has a vertex of degree not exceeding five.

**Corollary 3:** If $G$ is a connected planar simple graph with $e$ edges and $v$ vertices, where $v \geq 3$, and has no circuits of length 3, then $e \leq 2v - 4$. 
**Kuratowski's Theorem**

**Definition:** An *elementary subdivision* is an operation on a graph in which an edge \{u,v\} is removed, a new vertex \(w\) is added, and the edges \{u,w\} and \{w,v\} are added.
Kuratowski's Theorem

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**Definition:** Two graphs are said to be *homeomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.
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**Definition**: Two graphs are said to be *homeomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

**Kuratowski’s Theorem**: A graph is nonplanar if and only if it contains a subgraph homeomorphic to \(K_{3,3}\) or \(K_5\).
Graph Coloring

**Definition:** A *coloring* of a graph $G$ is an assignment that assigns a color to each vertex in $G$ such that no two adjacent vertices are assigned the same color.

**Definition:** A graph $G$ is said to be *$k$-colorable* if and only if $G$ can be colored using $k$ or fewer colors. The smallest value of $k$ for the graph $G$ is called the *chromatic number* of $G$ and is denoted by $\chi(G)$.
Graph Coloring: Theorems

**Theorem**: Any coloring of a graph with $n$ vertices uses at most $n$ colors.

**Theorem**: Any coloring of a complete graph $K_n$ uses at least $n$ colors.

**Corollary**: The chromatic number of a complete graph $\chi(K_n)$ is $n$.

**Theorem**: Any planar graph $G$ can be colored using $\max\text{-deg}(G) + 1$ colors.

**Theorem**: The chromatic number of a planar graph is no greater than 6.
The Four-Color Theorem

The Four-Color Theorem: The chromatic number of a planar graph is no greater than 4.