Problem 1

(10 points). Here’s a picture of an L-shaped triomino:

![L-shaped triomino](image)

Prove that you can use L-shaped triominoes to cover all but one tile on any chessboard of size $2^n \times 2^n$, where $n$ is a natural number.

Problem 2

(10 points).

(This one’s a very challenging, but very rewarding problem. Don’t hesitate to ask us for help if you’re stuck, but please mull it over for a bit on your own first!)

Two people, John and Karen, want to play a game. This game starts with a third person, Steve, writing two consecutive positive integers on two sticky notes, and then taping one note to John’s forehead and the other to Karen’s. Neither John nor Karen are told which number is taped to their own foreheads, but they can see which number is taped to the other person’s forehead. They also know the numbers are consecutive.

The game works like this: one player (say John, but it could be either) starts by asking the other player (Karen) if she knows her number. If she does, she states which number it is and the game ends. Otherwise, she says she doesn’t know, then asks John if he knows which number is on his forehead. If John knows, he states what his number is, or states that he doesn’t know and asks Karen the same question. The game continues until one of the players finally states his or her own number. John and Karen both have perfect reasoning ability, so if it is possible in some way for either of them to deduce what their number is, it is guaranteed that they will be able to do so.
Let \( n \) be the lesser of the two consecutive numbers. **Prove that this game always ends in at most \( 2n \) turns.**

a) Think about this problem on your own for 10 minutes before reading the following spoilers.

b) Assume John’s number is 1, and Karen’s number is 2. Karen plays first. Explain the sequence of events that leads to the end of the game.

c) Now assume John’s number is 2, and Karen’s number is 3. Karen plays first. Explain the sequence of events that leads to the end of the game.

d) Hopefully, you now have a better handle on the game, and have convinced yourself that it eventually **will** end. Now, prove the theorem above. Feel free to use any relevant insights from (b) or (c) in your proof.

**Problem 3**

(20 points each). Prove the following using Strong Induction:

a) Let \( a_n \) be the sequence defined by \( a_1 = 1, a_2 = 8, \) and \( a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3. \) Prove that \( a_n = 3(2^{n-1}) + 2(-1)^n \) for all \( n \in \mathbb{N}. \)

b) For any \( n \in \mathbb{Z}^+ \), prove that \( n \) is a perfect square (that is, \( n = k^2 \) for some \( k \in \mathbb{Z}^+ \)) if and only if \( n \) has an odd number of positive divisors. [Hint: if \( n \) is a perfect square, start by looking at its prime factorization.]

c) Prove that any natural number can be written as a sum of distinct powers of 2.

**Problem 4**

(10 points each).

a) Use the definition of Big-O notation to show that \( 3x^3 - 1000x - 200 \) is \( O(x^3). \)

b) Show that \( 3x^3 - 1000x - 200 \) is \( O(x^s) \) for all integers \( s > 3. \)