Practice Problems 2 ANWERS
CS 311: Discrete Math for CS (Bulko)

Problem 1

a) Let \( X \in \mathcal{P}(A \cap B) \). Then \( X \subset A \cap B \). So, \( X \subset A \) and \( X \subset B \). Therefore, \( X \in \mathcal{P}(A) \) and \( X \in \mathcal{P}(B) \) which implies \( X \in \mathcal{P}(A) \cap \mathcal{P}(B) \). This gives \( \mathcal{P}(A \cap B) \subset \mathcal{P}(A) \cap \mathcal{P}(B) \). Let \( Y \in \mathcal{P}(A) \cap \mathcal{P}(B) \). Then \( Y \in \mathcal{P}(A) \) and \( Y \in \mathcal{P}(B) \). So, \( Y \subset A \) and \( Y \subset B \). Therefore, \( Y \subset A \cap B \), which implies \( Y \in \mathcal{P}(A \cap B) \). This gives \( \mathcal{P}(A) \cap \mathcal{P}(B) \subset \mathcal{P}(A \cap B) \). Hence \( \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \).

b) Let \( a \) be an element of \( A \oplus B \). By the definition of symmetric difference, this means \( a \) is in \( A \) or \( a \) is in \( B \), but not in both \( A \) and \( B \). On the right hand side, if \( a \) is in \( (A \cup B) \), this means \( a \) is in \( A \) or \( a \) is in \( B \), and “subtracting” off the set \( (A \cap B) \) means that \( a \) is not in both \( A \) and \( B \). Thus, \( A \oplus B \) is a subset of \( (A \cup B) - (A \cap B) \). We can use a similar argument to prove that \( (A \cup B) - (A \cap B) \) is a subset of \( A \oplus B \). Thus, because the two sets are subsets of one another, they must be equal.

c. Let \( a \) be an element of \( A \oplus B \). By the definition of symmetric difference, this means \( a \) is in \( A \) or \( a \) is in \( B \), but not in both \( A \) and \( B \). On the right hand side, if \( a \) is in \( (A - B) \cup (B - A) \), this means \( a \) is in \( A \) but \( a \) is NOT in \( B \), or that \( a \) is in \( B \) but \( a \) is NOT in \( A \). This is exactly the definition of the symmetric difference, so \( A \oplus B \) is a subset of \( (A - B) \cup (B - A) \). We can use a similar argument to prove that \( (A - B) \cup (B - A) \) is a subset of \( A \oplus B \). Because the two sets are subsets of one another, they must be equal.

Problem 2

a) Suppose \( g \circ f(x) = g \circ f(y) = g(f(x)) = g(f(y)) = f(x) = f(y) \) as \( g \) is one-one. Thus, \( x = y \) as \( f \) is one-one. Hence, \( g \circ f \) is one-one.

b) Given an arbitrary element \( z \in C \), there exists a pre-image \( y \) of \( z \) under \( g \) such that \( g(y) = z \), since \( g \) is onto. Further, for \( y \in B \), there exists an element \( x \) in \( A \) with \( f(x) = y \), since \( f \) is onto. Therefore, \( g \circ f(x) = g(f(x)) = g(y) = z \), showing that \( g \circ f \) is onto.

c) Consider \( f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\} \) and \( g : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3\} \) defined as \( f(1) = 1, f(2) = 2, f(3) = f(4) = 3, g(1) = 1, g(2) = 2 \). Hence \( (g \circ f)(3) = g(4) = 3 \). It can be seen that \( g \circ f \) is onto but \( f \) is not onto.

Problem 3

Prove the following using mathematical induction:

a) Let \( P(n) \) be the given statement, i.e., \( P(n) : (1 + x)^n \geq (1 + nx) \), for \( x > -1 \). We note that \( P(n) \) is true when \( n = 1 \), since \((1 + x)(1 + x)\) for \( x > -1 \). Assume that \( P(k) : (1 + x)^k \geq (1 + kx) \), for \( x > -1 \) is true. We want to prove that \( P(k + 1) \) is true for \( x > -1 \). whenever \( P(k) \) is true.....

*\( (2) \) * Consider the identity \( (1 + x)^{k+1} = (1 + x)^k (1 + x) \). Given that \( x > -1 \), so \((1 + x) > 0 \). Therefore, by using \((1 + x)^k \geq (1 + kx) \), we have \((1 + x)^{k+1} \geq (1 + kx)(1 + x) \) i.e. \((1 + x)^{k+1} \geq (1 + x + kx + kx^2) \). Here \( k \) is a natural number and \( x^2 \geq 0 \) so that \( kx^2 \geq 0 \). Therefore \((1 + x + kx + kx^2) \geq (1 + x + kx) \), and so we obtain \((1 + x)^{k+1} \geq (1 + x + kx) \). i.e.
d) We will use strong induction to prove this result. Let $P(n)$ be formed using 5-cent and 6-cent stamps, for $n$ greater than or equal to 20.

\[(1 + x)^{k+1} \geq [1 + (1 + k)x].\] Thus the statement *(2)* is established. Hence by the principal of mathematical induction, $P(n)$ is true for all natural numbers.

b) Let the statement $P(n)$ be defined as $P(n) : 2(7^n) + 3(5^n) - 5$ is divisible by 24. We note that $P(n)$ is true for $n = 1$, since $2(7) + 3(5) - 5 = 24$, which is divisible by 24. Assume that $P(k)$ is true i.e. $2(7^k) + 3(5^k) - 5 = 24q$, when $k \in N$. Now, we wish to prove that $P(k + 1)$ is true whenever $P(k)$ is true. We have $2(7^{k+1}) + 3(5^{k+1}) - 5 = 2(7^k)(7) + 3(5^k)(5) - 5 = 7[2(7)^k + 3(5)^k - 5 - 3(5)^k + 5] + 3(5)^k(5) - 5 = 7[24q - 3(5)^k + 5] + 15(5)^k - 5 = 7(24q - 21(5)^k) + 35 + 15(5)^k - 5 = 7q + 24q - 6(5)^k + 30 = 724q - 6(5^k - 5) = 7q + 24q - 6(4p)[(5k - 5) is a multiple of 4]. = 7q + 24q - 24q = 24(7q - p) = 24 \times r; r = 7q - p$, is some natural number. Thus, divisible by 24. Hence, by principle of mathematical induction, $P(n)$ is true for all $n \in N$.

c) Let $P(n)$ be the given statement. i.e., $P(n) : 1^2 + 2^2 + ... + n^2 > n^3/3, n \in N$. We note that $P(n)$ is true for $n = 1$ since $1^2 > 1^2/3$. Assume that $P(k)$ is true i.e. $P(k) : 1^2 + 2^2 + ... + k^2 > k^3/3$. We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true. We have $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = (1^2 + 2^2 + 3^2 + ... + k^2) + (k + 1)^2 > k^3/3 + (k + 1)^2 = 1/3[k^3 + 3k^2 + 6k + 3] = 1/3[(k + 1)^3 + 3k + 2] > 1/3(k + 1)^3$. Therefore, $P(k + 1)$ is also true whenever $P(k)$ is true. Hence, by mathematical induction $P(n)$ is true for all $n \in N$.

d) We will use strong induction to prove this result. Let $P(n)$ be the statement that postage of $n$ cents can be formed using 5-cent and 6-cent stamps, for $n$ greater than or equal to 20.

- Base case: The propositions $P(20), P(21), P(22), P(23)$ and $P(24)$ are true because $20 = 5 + 5 + 5 + 5$, $21 = 5 + 5 + 5 + 6$, $22 = 5 + 5 + 6 + 6$, $23 = 5 + 6 + 6 + 6$, and $24 = 6 + 6 + 6 + 6$.
- Inductive hypothesis: The inductive hypothesis is the statement that $P(j)$ is true for $20 \leq j \leq k$ where $k$ is an integer with $k \geq 24$.
- Inductive step: we need to show that $P(k + 1)$ is true. Using the inductive hypothesis we can assume that $P(k - 4)$ is true because $k - 4$ must be at least 20. Hence we can form postage of $k - 4$ cents using 5-cent and 6-cent stamps. Just add one extra 5-cent stamp to the solution for $k - 4$ cents and get postage for $k + 1$ cents. Hence $P(k + 1)$ is true.

**Problem 4**

a) We need to sort the functions in decreasing order of growth rate. The order is as follows:

\[(n!)^2, 10^n, n^{100}, n^{99} + n^{99}, \sqrt{n} \log n, (\log n)^3\]

b) 

1. $O(n^3 \log n)$
2. $O((2^n)(3^n))$