Formal and Informal Proofs
Logical Arguments

Definitions:

- An *argument* in propositional logic is a sequence of propositions.
- All but the final proposition in the argument are called *premises*.
- The final proposition is called the *conclusion*.
- An argument is *valid* if the truth of all of its premises implies that the conclusion is true.

Note that if an argument with premises $p_1, p_2, \ldots, p_n$ and conclusion $q$ is valid, then

\[(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q\]

is a tautology. The above form is called the *tautological form* of the argument.
Formal Proofs

A *formal proof* of a statement:

- provides an argument supporting the validity of the statement
- shows that the conclusion follows from the premises
- uses Premises, Axioms, and results of other theorems
- show that each step of the proof follows logically from the Premises, Axioms, and other theorems.

In order to show that each step "follows logically", we use *Rules of Inference*.

Rules of Inference:

- represent logically valid *argument forms* or inference patterns
- allow us to infer new True statements from existing True statements
Informal Proofs

In *informal proofs*:

- The steps of the proofs are not expressed in any formal language such as propositional logic.
- Steps are argued less formally using English, mathematical formulas, etc.
- Logic and Rules of Inference are used to help us decide whether or not each step of the argument is sound.
Methods of Proving Theorems

Direct proof:
- $p \rightarrow q$ is proved by showing that if $p$ is True, $q$ follows.

Indirect proof:
- to prove $p \rightarrow q$, prove the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds, $\neg p$ follows.

Proof by contradiction:
- show that $(p \land \neg q)$ contradicts the assumptions

Proof by cases
- break the proof into cases and prove each one individually

Proof of equivalences:
- $(p \leftrightarrow q)$ is proved by proving $(p \rightarrow q) \land (q \rightarrow p)$. 
Proofs with Quantifiers

Existential Quantifiers:
1. Find an example (by making educated guesses) that shows that the statement holds.
2. Prove by contradiction: negate the existentially quantified statement and show that it implies a contradiction.
3. To disprove: choose an arbitrary $x$ and show that no matter what $x$ is, the property is False.

Universal Quantifiers:
1. Choose an arbitrary $x$ and show that no matter what $x$ is, the property holds true.
2. Proof by Cases: split the statement into cases and prove each of them.
3. Find a counterexample.