Functions
Functions

Definition: Let A and B be sets. A function from A to B, denoted \( f : A \rightarrow B \), is an assignment of exactly one element of B to each element of A. We write \( f(a) = b \) to denote the assignment of \( b \) to \( a \) by the function \( f \), where \( a \in A \) and \( b \in B \).
Important Sets Associated with Functions

**Definitions:** Let $f$ be a function from $A$ to $B$. Let $f(a) = b$ for $a \in A$ and $b \in B$.

- $A$ is the *domain* of $f$ and $B$ is the *codomain* of $f$. (Why not "range"?)
- $b$ is the *image* of $a$, and $a$ is a *pre-image* of $b$. (Why not "the"?)
- The *range* of $f$ is the set of all images of elements of $A$. That is, the range of $f = \{ b \in B \mid \forall a \in A, f(a) = b \}$.
- We also say $f$ *maps* $A$ to $B$.

**Definition:** Let $f$ be function from $A$ to $B$. Let $S$ be a subset of $A$. The *image* of $S$ is a subset of $B$ that consists of the images of the elements of $S$. We denote the image of $S$ by $f(S)$, so that $f(S) = \{ f(s) \mid s \in S \}$.

Note that the first definition of "image" defines the image of a single element, and the second definition defines the image of a set.
**Important Properties of Functions**

**Definition:** A function $f$ is said to be *one-to-one*, or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y$ in the domain of $f$. A function is said to be an *injection* if it is one-to-one.

Alternate definition: A function is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. (Note that this is the contrapositive of the other definition.)

**Definition:** A function $f$ is said to be *onto*, or *surjective*, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. It is also said that set $B$ is *covered* by $f$.

**Definition:** A function if said to be *bijective* (or called a *bijection*) if it is both one-to-one and onto.
**Theorem**

**Theorem**: Let $f$ be a function $f: A \rightarrow A$ from a finite set $A$ to itself. Then $f$ is one-to-one if and only if $f$ is onto.
Functions on Real Numbers

**Definition:** Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$ (real numbers). Then $f_1 + f_2$ and $f_1 * f_2$ are also functions from $A$ to $\mathbb{R}$, defined by:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 * f_2)(x) = f_1(x) * f_2(x)$$

**Definition:** A function $f$ whose domain and codomain are subsets of $\mathbb{R}$ is *strictly increasing* if, for $x$ and $y$ in the domain of $f$, $f(x) > f(y)$ whenever $x > y$. Similarly, $f$ is *strictly decreasing* if $f(x) < f(y)$ whenever $x > y$.

**Definition:** Let $A$ be a set. The *identity function* on $A$ is the function $i_A : A \rightarrow A$ where $i_A(x) = x$. 
Definition: Let $f$ be a bijection from set $A$ to set $B$. The inverse function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$. Hence, $f^{-1}(b) = a$, when $f(a) = b$. If the inverse function of $f$ exists, $f$ is called invertible.
Composition of Functions

**Definition:** Let $f$ be a function from set $A$ to set $B$, and let $g$ be a function from set $B$ to set $C$. The *composition* of the functions $g$ and $f$, denoted by $(g \circ f)(a) = g(f(a))$. 