Sequences and Summations
Sequences

**Definition:** A *sequence* is a function from a subset of the set of integers (most commonly the whole numbers or positive integers) to a set $S$.

- We use the notation $a_n$ to denote the image of the integer $n$.
- We call $a_n$ a *term* of the sequence.
- We use the notation $\{a_n\}$ to describe the sequence.
Definition: An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd$$

where $a$ (called the initial term) and $d$ (called the common difference) are real numbers.

Definition: A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^k$$

where $a$ (called the initial term) and $r$ (called the common ratio) are real numbers.
Recurrence Relations

**Definition:** A *recurrence relation* for the sequence \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more of the previous terms of the sequence \( a_0, a_1, a_2, \ldots, a_{n-1} \) for all \( n \geq n_0 \), where \( n_0 \) is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- A recurrence relation is said to *recursively define* a sequence.
- The *initial conditions* for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.

**Definition:** When a recurrence relation can be described with an explicit formula, we say that we have *solved* the recurrence relation by finding a *closed* formula.
Some useful sequences

| n^2   | 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. . . |
| n^3   | 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000. . . |
| n^4   | 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000. . . |
| 2^n   | 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. . . |
| 3^n   | 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049. . . |
| n!    | 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800. . . |
| fib_n | 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89. . . |
Summations

Summation of the terms of a sequence

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n \]

- The variable j is referred to as the *index of summation*.
- m is the *lower limit*.
- n is the *upper limit*. 
Arithmetic Series

**Definition**: The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is called an arithmetic series.

**Theorem**: The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is:

\[
S = \sum_{j=0}^{n} (a + jd) = a(n+1) + d \frac{n(n+1)}{2}
\]
Geometric Series

**Definition**: The sum of the terms of the geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is called a geometric series.

**Theorem**: The sum of the terms of the geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is:

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]$$