Matrices
Basic Definitions

**Definition:** A *matrix* is a rectangular array of numbers.

**Definition:** A matrix with m rows and n columns is called an $m \times n$ *matrix*.

**Definition:** A matrix with the same number of rows as columns is called a *square matrix*.

**Definition:** Two matrices are *equal* if they have the same number of rows, the same number of columns, and the corresponding entries in every position are equal.
**Definition:** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The sum of $A$ and $B$, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its $(i,j)$th element. In other words, $A + B = [a_{ij} + b_{ij}]$.

The sum is not defined if the two matrices are of different size.
Matrix Multiplication

**Definition:** Let $A$ be an $m \times k$ matrix and $B$ a $k \times n$ matrix. The *product* of $A$ and $B$, denoted by $AB$, is the $m \times n$ matrix that has its $(i,j)$th element equal to the sum of the products of the corresponding elements from the $i$th row of $A$ and the $j$th column of $B$. In other words, if $AB = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}.$$ 

The product is not defined when the number of columns in the first matrix is not equal to the number of rows in the second matrix.
Identity Matrix

**Definition:** The *identity matrix of order* $n$ is the $n \times n$ matrix $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. 

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$
Powers of Square Matrices

**Definition:** When $A$ is an $n \times n$ matrix, we define:

\[
A^0 = I_n
\]

\[
A^r =AAAA\ldots AA
\quad (r \text{ times})
\]
Transpose of a Matrix

**Definition:** Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $A$, denoted by $A^T$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $A$.

If $A^T = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m$. 
**Inverse of a Matrix**

**Definition:** Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *inverse* of $A$, denoted by $A^{-1}$, is the $n \times n$ matrix such that $AA^{-1} = A^{-1}A = I_n$

**Note:** a matrix may not have an inverse.
Symmetric Matrices

**Definition:** A square matrix $A$ is called *symmetric* if $A = A^T$.

$A = [A_{ij}]$ is symmetric if $a_{ij} = b_{ij}$ for $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m$. 
Zero-one Matrices

**Definition**: A matrix with entries that are all either 0 or 1 is called a *zero-one matrix*.

**Definition**: Let $A$ and $B$ be two zero-one matrices.

The *join* of $A$ and $B$ is the zero-one matrix with $(i,j)$th entry $a_{ij} \lor b_{ij}$. The join of $A$ and $B$ is denoted by $A \lor B$.

The *meet* of $A$ and $B$ is the zero-one matrix with $(i,j)$th entry $a_{ij} \land b_{ij}$. The join of $A$ and $B$ is denoted by $A \land B$. 

**Boolean Product**

**Definition:** Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix, and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix. Then the *Boolean product* of $A$ and $B$, denoted by $A \odot B$, is the $m \times n$ matrix with $(i,j)$th entry $c_{ij}$, where

$$c_{ij} = (a_{i1} \land b_{1j}) \lor (a_{i2} \land b_{2j}) \lor \ldots \lor (a_{ik} \land b_{kj}).$$

The Boolean product is not defined when the number of columns in the first matrix is not equal to the number of rows in the second matrix.