Integers

C Puzzles

```c
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

Assume a machine with 32-bit, two’s complement integers.

For each of the following, either:
- Argue that is true for all argument values;
- Give an example where it’s not true.

- \( x < 0 \) → \((x*2) < 0\)
- \( ux >= 0 \) ➔ \((x<<7) < 0\)
- \( (x & 7) == 7 \) ➔ \((x<<30) < 0\)
- \( ux > -1 \)
- \( x > y \) ➔ \(-x < -y\)
- \( x * x >= 0 \)
- \( x > 0 && y > 0 \) ➔ \(x + y > 0\)
- \( x >= 0 \) ➔ \(-x <= 0\)
- \( x <= 0 \) ➔ \(-x >= 0\)

Encoding Integers: Unsigned

For unsigned integers, we treat all values as non-negative and use positional notation as with non-negative decimal numbers.

Assume we have a \( w \) length bit string \( X \).

\[
\text{Unsigned: } B2U_w(X) = \sum_{i=0}^{w-1} X_i \times 2^i
\]
Encoding Integers: Two's Complement

Two's complement is a way of encoding integers, including some positive and negative values. It's exactly like unsigned except the high order bit is given negative weight.

Two's complement: \( B_2T_w(X) = -X_{w-1} \times 2^{w-1} + \sum_{i=0}^{w-2} X_i \times 2^i \)

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Hex</th>
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</tr>
</thead>
<tbody>
<tr>
<td>15213</td>
<td>3B 6D</td>
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</tr>
<tr>
<td>-15213</td>
<td>C4 93</td>
<td>11000100 10010011</td>
</tr>
</tbody>
</table>

Sign Bit:
For 2's complement, the most significant bit indicates the sign.
- 0 for nonnegative
- 1 for negative

Values for Different Word Sizes

<table>
<thead>
<tr>
<th>( w )</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMax</td>
<td>255</td>
<td>65,525</td>
<td>4,294,967,295</td>
<td>18,446,744,073,709,551,615</td>
</tr>
<tr>
<td>TMax</td>
<td>127</td>
<td>32,767</td>
<td>2,147,483,647</td>
<td>9,223,372,036,854,775,807</td>
</tr>
<tr>
<td>TMin</td>
<td>-128</td>
<td>-32,768</td>
<td>-2,147,483,648</td>
<td>-9,223,372,036,854,775,808</td>
</tr>
</tbody>
</table>

Observations
- \(|T\text{Min}| = T\text{Max} + 1\)
- \(U\text{Max} = 2 \times T\text{Max} + 1\)

C Programming

```
#include <limits.h>
```

Declares various constants: \(U\text{LONG\_MAX}, L\text{ONG\_MAX}, L\text{ONG\_MIN}, \) etc. The values are platform-specific.
Unsigned and Signed Numeric Values

**Equivalence:** Same encoding for nonnegative values

**Uniqueness:**
- Every bit pattern represents a unique integer value
- Each representable integer has unique encoding

**Can Invert Mappings:**
- inverse of B2U(X) is U2B(X)
- inverse of B2T(X) is T2B(X)

<table>
<thead>
<tr>
<th>X</th>
<th>B2U(X)</th>
<th>B2T(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>1001</td>
<td>9</td>
<td>-7</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>-6</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>-5</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>-4</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>-2</td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td>-1</td>
</tr>
</tbody>
</table>

C allows conversions from signed to unsigned.

```c
short int x = 15213;
unsigned short ux = (unsigned short) x;
short int y = -15213;
unsigned short uy = (unsigned short) y;
```

**Resulting Values:**
- The bit representation stays the same.
- Nonnegative values are unchanged.
- Negative values change into (large) positive values.

Signed vs Unsigned in C

**Constants**
- By default, constants are considered to be signed integers.
- They are unsigned if they have "U" as a suffix: 0U, 4294967259U.

**Casting**
- Explicit casting between signed and unsigned is the same as U2T and T2U:
  ```c
  int tx, ty;
  unsigned ux, uy;
  tx = (int) ux;
  uy = (unsigned) ty;
  ```
- Implicit casting also occurs via assignments and procedure calls.
  ```c
  tx = ux;
  uy = ty;
  ```

**Expression Evaluation**
- If you mix unsigned and signed in a single expression, signed values implicitly cast to unsigned.
- This includes when you compare using <, >, ==, <=, >=.

<table>
<thead>
<tr>
<th>Const 1</th>
<th>Const 2</th>
<th>Rel.</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0U</td>
<td>==</td>
<td>unsigned</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>&lt;</td>
<td>signed</td>
</tr>
<tr>
<td>-1</td>
<td>0U</td>
<td>&gt;</td>
<td>unsigned</td>
</tr>
<tr>
<td>2147483647</td>
<td>-2147483648</td>
<td>&gt;</td>
<td>signed</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>&gt;</td>
<td>signed</td>
</tr>
<tr>
<td>(unsigned) 1</td>
<td>-2</td>
<td>&lt;</td>
<td>unsigned</td>
</tr>
<tr>
<td>2147483647</td>
<td>2147483648U</td>
<td>&lt;</td>
<td>unsigned</td>
</tr>
<tr>
<td>2147483647</td>
<td>(int) 2147483648U</td>
<td>&gt;</td>
<td>signed</td>
</tr>
</tbody>
</table>
**Sign Extension**

**Task:** Given a w-bit signed integer x, convert it to a w+k-bit integer with the same value.

**Rule:** Make k copies of the sign bit:

\[ x' = x_{w-1}, \ldots, x_{w-1}, x_{w-2}, \ldots, x_0 \]

Why does this work?

- zero-extend extends unsigned values to wider mode
- sign-extend extends signed values to wider mode

**Sign Extension Example**

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>15213</td>
<td>3B 6D 00111011 01101101</td>
</tr>
<tr>
<td>ix</td>
<td>15213</td>
<td>00 00 3B 6D 00000000 00000000 00111011 01101101</td>
</tr>
<tr>
<td>y</td>
<td>-15213</td>
<td>C4 93 11000100 10010011</td>
</tr>
<tr>
<td>iy</td>
<td>-15213</td>
<td>FF FF C4 93 11111111 11111111 11000100 10010011</td>
</tr>
</tbody>
</table>

In converting from smaller to larger signed integer data types, C automatically performs sign extension.

**Why Use Unsigned?**

Don’t use just to ensure numbers are nonzero.

- Some C compilers generate less efficient code for unsigned.

```c
unsigned i;
for (i=1; i < cnt; i++)
a[i] += a[i-1]
```

- It’s easy to make mistakes.

```c
for (i = cnt-2; i >= 0; i--)
a[i] += a[i+1]
```

Do use when performing modular arithmetic.

- multiprecision arithmetic
- other esoteric stuff

Do use when you need extra bits of range.

**Negating Two’s Complement**

To find the negative of a number in two’s complement form: complement the bit pattern and add 1:

\[ \sim x + 1 = -x \]

**Example:**

10011101 = 0x9C = -99\text{10}

complement:

01100010 = 0x62 = 98\text{10}

add 1:

01100011 = 0x63 = 99\text{10}

Try it with: 11111111 and 00000000. 
### Complement and Increment Examples

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>15213</td>
<td>00111011 01101101</td>
</tr>
<tr>
<td>~x</td>
<td>-15214</td>
<td>11000100 10010010</td>
</tr>
<tr>
<td>~x+1</td>
<td>-15213</td>
<td>11000100 10010011</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>00 00 00000000 00000000</td>
</tr>
<tr>
<td>~0</td>
<td>-1</td>
<td>FF FF 11111111 11111111</td>
</tr>
<tr>
<td>~0+1</td>
<td>0</td>
<td>00 00 00000000 00000000</td>
</tr>
</tbody>
</table>

### Unsigned Addition

Given two w-bit unsigned quantities u, v, the true sum may be a w+1-bit quantity.

**Discard the carry bit** and treat the result as an unsigned integer.

Thus, unsigned addition implements **modular addition**.

\[
\text{UAdd}_w(u, v) = (u + v) \mod 2^w
\]

\[
\text{UAdd}_w(u, v) = \begin{cases} 
  u + v & u + v < 2^w \\
  u + v - 2^w & u + v \geq 2^w 
\end{cases}
\]

### Detecting Unsigned Overflow

**Task:**

Determine if \( s = \text{UAdd}_w(u, v) = u + v \).

**Claim:** We have overflow iff:

\[
s < u \text{ or } s < v.
\]

**BTW:** \( s < u \text{ iff } s < v \). So it’s OK to check only one of these conditions because both will be true when there’s an overflow.

On the machine, this causes the **carry flag** to be set.

### Properties of Unsigned Addition

W-bit unsigned addition is:

- **Closed under addition:**
  \[
  0 \leq \text{UAdd}_w(u, v) \leq 2^w - 1
  \]

- **Commutative**
  \[
  \text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)
  \]

- **Associative**
  \[
  \text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)
  \]

- **0 is the additive identity**
  \[
  \text{UAdd}_w(u, 0) = u
  \]

- **Every element has an additive inverse**
  Let \( \text{UComp}_w(u) = 2^w - u \), then
  \[
  \text{UAdd}_w(u, \text{UComp}_w(u)) = 0
  \]
Given two w-bit signed quantities $u$, $v$, the true sum may be a $w+1$-bit quantity.

Discard the carry bit and treat the result as a two’s complement number.

$$TAdd_w(u,v) = \begin{cases} 
    u + v + 2^w & u + v < TMin_w \text{ (NegOver)} \\
    u + v & TMin_w < u + v \leq TMax_w \\
    u + v - 2^w & TMax_w < u + v \text{ (PosOver)}
\end{cases}$$

TAdd and UAdd have identical bit-level behavior.

```c
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```
This will give $s == t$.

Detecting 2’s Complement Overflow

Task: Determine if $s = TAdd_w(u,v) = u + v$.

Claim: We have overflow iff either:
- $u, v < 0$ but $s \geq 0$ (NegOver)
- $u, v \geq 0$ but $s < 0$ (PosOver)

Can compute this as:

```c
ovf = (u<0 == v<0) && (u<0 != s<0);
```
On the machine, this causes the overflow flag to be set.

Why don’t we have to worry about the case where one input is positive and one negative?

Properties of TAdd

TAdd is Isomorphic to UAdd.
This is clear since they have identical bit patterns.

$$Tadd_w(u,v) = U2T(UAdd_w(T2U(u), T2U(v)))$$

Two’s Complement under TAdd forms a group.
- Closed, commutative, associative, 0 is additive identity.
- Every element has an additive inverse:

Let $TComp_w(u) = U2T(UComp_w(T2U(u)))$, then $TAdd_w(u, UComp_w(u)) = 0$

$$TComp_w(u) = \begin{cases} 
    -u & u \neq TMin_w \\
    TMin_w & u = TMin_w
\end{cases}$$
Computing the exact product of two \( w \)-bit numbers \( x, y \). This is the same for both signed and unsigned.

**Ranges:**
- **Unsigned:** \( 0 \leq x \times y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1 \), requires up to \( 2w \) bits.
- **Two’s comp. min:**
  \[ x \times y \geq (-2^{w-1})(2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}, \]
  requires up to \( 2w - 1 \) bits.
- **Two’s comp. max:** \( x \times y \leq (-2^{w-1})^2 = 2^{2w-2} \), requires up to \( 2w \) (but only for \( TMin^2_w \)).

**Maintaining the exact result**
- Would need to keep expanding the word size with each product computed.
- Can be done in software with “arbitrary precision” arithmetic packages.

**Unsigned Multiplication in C**

Given two \( w \)-bit unsigned quantities \( u, v \), the true sum may be a \( 2w \)-bit quantity.

We just discard the most significant \( w \) bits, treat the result as an unsigned number.

Thus, unsigned multiplication implements **modular multiplication**.

\[
UMult_w(u, v) = (u \times v) \mod 2^w
\]

---

**Unsigned vs. Signed Multiplication**

**Unsigned Multiplication**

\[
\begin{align*}
\text{unsigned } ux &= (\text{unsigned}) x; \\
\text{unsigned } uy &= (\text{unsigned}) y; \\
\text{unsigned } up &= ux \times uy;
\end{align*}
\]

- Truncates product to \( w \)-bit number: \( up = UMult_w(ux, uy) \)
- Modular arithmetic: \( up = (ux \cdot uy) \mod 2^w \)

**Two’s Complement Multiplication**

\[
\begin{align*}
\text{int } x, y; \\
\text{int } p &= x \times y;
\end{align*}
\]

- Compute exact product of two \( w \)-bit numbers \( x, y \).
- Truncate result to \( w \)-bit number: \( p = TMult_w(x, y) \)

**Relation**
- Signed multiplication gives same bit-level result as unsigned.
- \( up == (\text{unsigned}) p \)
Multiply with Shift

A left shift by \( k \), is equivalent to multiplying by \( 2^k \). This is true for both signed and unsigned values.

\[
\begin{align*}
u << 1 & \rightarrow u \times 2 \\
u << 2 & \rightarrow u \times 4 \\
u << 3 & \rightarrow u \times 8 \\
u << 4 & \rightarrow u \times 16 \\
u << 5 & \rightarrow u \times 32 \\
u << 6 & \rightarrow u \times 64 \\
\end{align*}
\]

Compilers often use shifting for multiplication, since shift and add is much faster than multiply (on most machines).

\[
u << 5 - u << 3 == u \times 24
\]

Aside: Floor and Ceiling Functions

Two useful functions on real numbers are the floor and ceiling functions.

**Definition:** The floor function \( \lfloor r \rfloor \), is the greatest integer less than or equal to \( r \).

\[
\begin{align*}
\lfloor 3.14 \rfloor &= 3 \\
\lfloor -3.14 \rfloor &= -4 \\
\lfloor 7 \rfloor &= 7
\end{align*}
\]

**Definition:** The ceiling function \( \lceil r \rceil \), is the smallest integer greater than or equal to \( r \).

\[
\begin{align*}
\lceil 3.14 \rceil &= 4 \\
\lceil -3.14 \rceil &= -3 \\
\lceil 7 \rceil &= 7
\end{align*}
\]

Unsigned Divide by Shift

A right shift by \( k \), is (approximately) equivalent to dividing by \( 2^k \), but the effects are different for the unsigned and signed cases.

**Quotient of unsigned value by power of 2.**

\[
u >> k == \lfloor u/2^k \rfloor
\]

Uses logical shift.

<table>
<thead>
<tr>
<th>Division</th>
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<th>Binary</th>
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<tbody>
<tr>
<td>( u )</td>
<td>15213</td>
<td>15213</td>
<td>3B 6D 00111011 01101101</td>
</tr>
<tr>
<td>( u &gt;&gt; 1 )</td>
<td>7606.5</td>
<td>7606</td>
<td>1D B6 00011101 10110110</td>
</tr>
<tr>
<td>( u &gt;&gt; 4 )</td>
<td>950.8125</td>
<td>950</td>
<td>03 B6 00000011 10110110</td>
</tr>
<tr>
<td>( u &gt;&gt; 8 )</td>
<td>59.4257813</td>
<td>59</td>
<td>00 3B 00000000 00111011</td>
</tr>
</tbody>
</table>

Signed Divide by Shift

**Quotient of signed value by power of 2.**

\[
u >> k == \lfloor u/2^k \rfloor
\]

- Uses arithmetic shift. What does that mean?
- Rounds in wrong direction when \( u < 0 \).

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<td>-15213</td>
<td>C4 93 11000100 10010011</td>
</tr>
<tr>
<td>( u &gt;&gt; 1 )</td>
<td>-7606.5</td>
<td>-7607</td>
<td>E2 49 11100010 01001001</td>
</tr>
<tr>
<td>( u &gt;&gt; 4 )</td>
<td>-950.8125</td>
<td>-951</td>
<td>FC 49 11111100 01001001</td>
</tr>
<tr>
<td>( u &gt;&gt; 8 )</td>
<td>-59.4257813</td>
<td>-60</td>
<td>FF C4 11111111 11000100</td>
</tr>
</tbody>
</table>
Correct Power-of-2 Division

We’ve seen that right shifting a negative number gives the wrong answer because it rounds away from 0.

\[ x \gg k = \lfloor \frac{x}{2^k} \rfloor \]

We’d really like \[ \lceil \frac{x}{2^k} \rceil \] instead.

You can compute this as: \[ \lfloor (x + 2^k - 1)/2^k \rfloor \]. In C, that’s:

\[
(x + (1<<k) - 1) \gg k
\]

This biases the dividend toward 0.

Properties of Unsigned Arithmetic

Unsigned multiplication with addition forms a **Commutative Ring**.

- Addition is commutative
- Closed under multiplication
  \[ 0 \leq \text{UMult}_w(u, v) \leq 2^w - 1 \]
- Multiplication is commutative
  \[ \text{UMult}_w(u, v) = \text{UMult}_w(v, u) \]
- Multiplication is associative
  \[ \text{UMult}_w(t, \text{UMult}_w(u, v)) = \text{UMult}_w(\text{UMult}_w(t, u), v) \]
- 1 is the multiplicative identity
  \[ \text{UMult}_w(u, 1) = u \]
- Multiplication distributes over addition
  \[ \text{UMult}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UMult}_w(t, u), \text{UMult}_w(t, v)) \]

Properties of Two’s Complement Arithmetic

**Isomorphic Algebras**

- Unsigned multiplication and addition: truncate to \( w \) bits
- Two’s complement multiplication and addition: truncate to \( w \) bits

**Both form rings isomorphic to ring of integers mod \( 2^w \)**

**Comparison to Integer Arithmetic**

- Both are rings
- Integers obey ordering properties, e.g.
  \[ u > 0 \rightarrow u + v > v \]
  \[ u > 0, v > 0 \rightarrow u \cdot v > 0 \]
- These properties are not obeyed by two’s complement arithmetic.
  \[ TMax + 1 == \text{TMin} \]
  \[ 15213 \times 30426 == -10030 \] (for 16-bit words)

C Puzzle Answers

Assume a machine with 32-bit word size, two’s complement integers.

```c
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- \( x < 0 \rightarrow (x*2) < 0 \)
  - False: TMin
- \( ux >= 0 \)
  - True: 0 = UMin
- \( (x & 7) == 7 \)
  - True: \( x_1 = 1 \)
- \( ux > -1 \)
  - False: 0
- \( x > y \)
  - \( -x < -y \)
  - False: \(-1, \text{TMin}\)
- \( x * x >= 0 \)
  - \( x + y > 0 \)
  - False: 30426
- \( x >= 0 \)
  - \( -x <= 0 \)
  - False: TMax, TMax
- \( x <= 0 \)
  - \( -x >= 0 \)
  - True: -TMax < 0