Fourier Series Formalization in ACL2(r)

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Outline

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Fourier series have many applications to a wide variety of mathematical and physical problems, electrical engineering, signal processing, etc.

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We are interested in formalizing Fourier series (and possibly, Fourier transform) in ACL2 as a useful tool for formally analyzing analog circuits, mixed-signal integrated circuits, etc.

Overview

In this work, we present our efforts in formalizing some basic properties of Fourier series in the logic of ACL2(r), which is a variant of ACL2 that supports reasoning about the real numbers by way of non-standard analysis [R. Gamboa, 1999].

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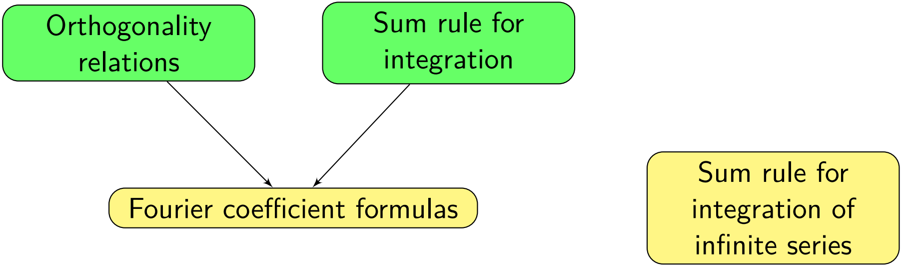

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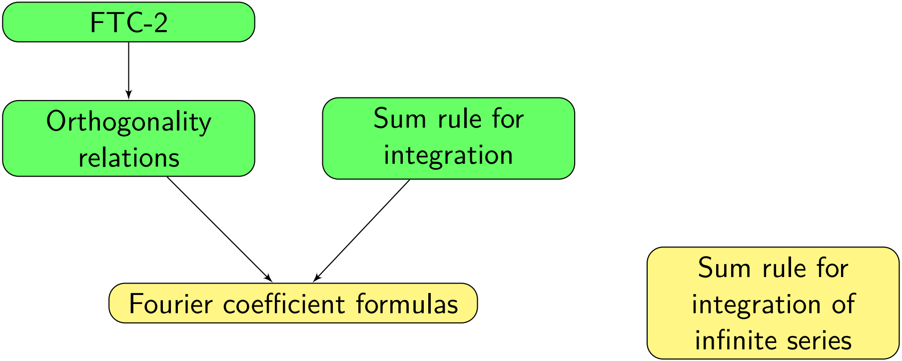

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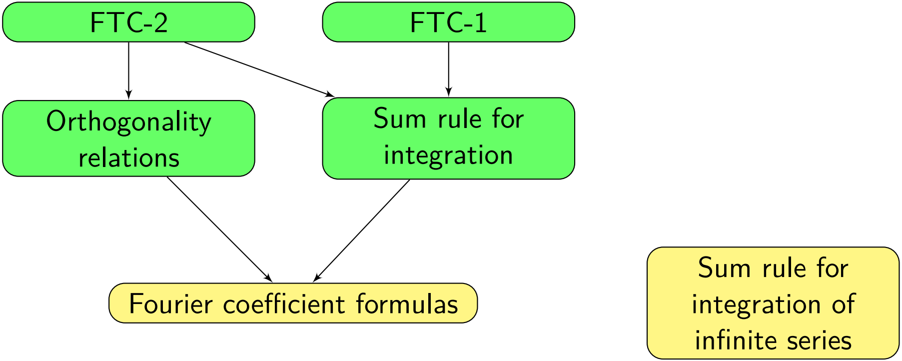

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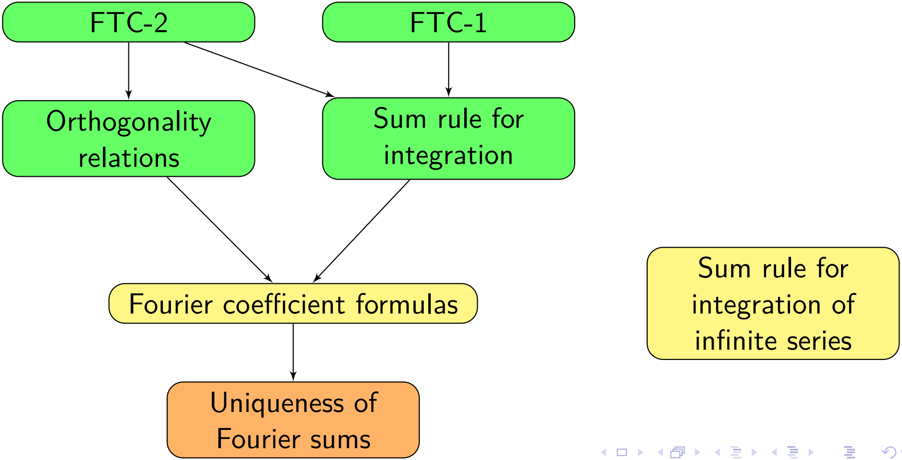

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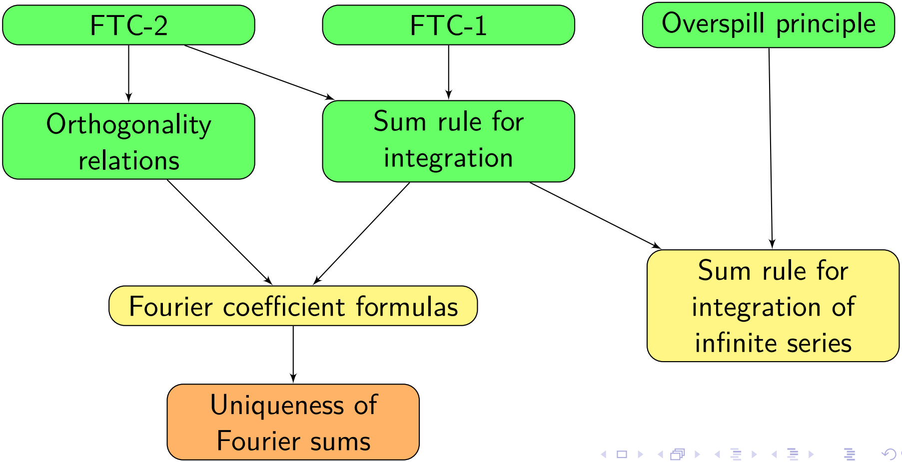

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Formulate the operations of calculus using a logically rigorous notion of infinitesimal numbers, instead of epsilon-delta definition of limit.

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Two basic approaches to the foundations:

1. Extend the reals to a bigger set of hyperreals, which includes infinitesimals [A. Robinson, 1996].
2. Nelson's Internal Set Theory views the "reals" as "all the reals", including infinitesimals, and considers a subset of standard reals [E. Nelson, 1977].

ACL2(r) follows (2).

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- Cool and fun!!!

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All terms introduced here are considered non-classical.

FTC-2 Evaluation Procedure

Cowles and Gamboa [J. Cowles & R. Gamboa, 2014] implemented a framework for evaluating definite integrals of real-valued continuous unary functions on a closed and bounded interval using the Second Fundamental Theorem of Calculus (FTC-2).

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We extend this framework to functions containing free argument(s) and call the extended framework the FTC-2 evaluation procedure.

$$\int_a^b f_1(x, n)dx = g_1(b, n) - g_1(a, n),$$

where $g_1'(x, n) = f_1(x, n), \forall x \in [a, b]$.

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The two following conditions must be satisfied in order to make such a substitution valid:

1. The new function symbols satisfy the constraints on the replaced function symbols.
2. Since (1) is a classical theorem, free variables are allowed to appear in the functional substitution as long as classicalness is preserved [R. Gamboa & J. Cowles, 2007].

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  Specifying an antiderivative of a function via a computer algebra system such as Mathematica [Wolfram Research, Inc., 2015].

  Proving the correctness of the antiderivative via the automatic differentiator (AD) implemented in ACL2(r) by Reid and Gamboa [P. Reid & R. Gamboa, 2011].

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Fourier Coefficient Formulas

Theorem 1 (Fourier coefficient formulas)

Consider the following Fourier sum for a periodic function with period $2L$:

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\tfrac{\pi}{L}x) + b_n \sin(n\tfrac{\pi}{L}x)\right)$$

Then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(m\frac{\pi}{L}x)dx,$$

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Lemma 2 (Sum rule for definite integrals of indexed sums)

Let $\{f_n\}$ *be a set of real-valued continuous functions on* $[a, b]$, *where* $n = 0, 1, 2, ..., N$. *Then*

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Prove by applying FTC-1, FTC-2, and the sum rule for differentiation.

Orthogonality Relations of Trigonometric Functions

Lemma 3 (Orthogonality relations of trigonometric functions)

$$\int_{-L}^{L} \sin(m\frac{\pi}{L}x)\sin(n\frac{\pi}{L}x)dx = \begin{cases} 0, & \textit{if } m \neq n \vee m = n = 0 \\ L, & \textit{if } m = n \neq 0 \end{cases}$$

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Prove by applying the FTC-2 evaluation procedure.

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Fourier coefficients of periodic functions are then formalized from the sum rule for integration (Lemma 2) and the orthogonality relations (Lemma 3).

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Uniqueness of Fourier Sums

Consequently, the uniqueness of Fourier sums is a straightforward corollary of the Fourier coefficient formulas (Theorem 1).

Corollary 4 (Uniqueness of Fourier sums)

Let

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

and

$$g(x) = A_0 + \sum_{n=1}^{N} (A_n \cos(n\frac{\pi}{L}x) + B_n \sin(n\frac{\pi}{L}x))$$

$$\text{Then } f = g \Leftrightarrow \begin{cases} a_0 = A_0 \\ a_n = A_n, \text{ for all } n = 1, 2, ..., N \\ b_n = B_n, \text{ for all } n = 1, 2, ..., N \end{cases}$$

Inner Product Formula

Our framework can also be applied to prove other Fourier series' properties, e.g., the following inner product formula (not presented in the paper):

Theorem 5 (Inner product formula)

Let

$$f(x) = a_0 + \sum_{n=1}^{M} (a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

and

$$g(x) = A_0 + \sum_{n=1}^{N} (A_n \cos(n\frac{\pi}{L}x) + B_n \sin(n\frac{\pi}{L}x))$$

Then

$$\frac{1}{L}\int_{-L}^{L} f(x)g(x)dx = 2a_0A_0 + \sum_{n=1}^{\min\{M,N\}} a_nA_n + \sum_{n=1}^{\min\{M,N\}} b_nB_n$$

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In non-standard analysis,

$$\int_a^b \mathrm{st}\left(\sum_{n=0}^{H_0} f_n(x)\right)dx \stackrel{?}{=} \mathrm{st}\left(\sum_{n=0}^{H_1}\int_a^b f_n(x)dx\right)$$

for all infinitely large natural numbers H_0 and H_1,
where st is the standard-part function in non-standard analysis.

Pointwise Convergence vs. Uniform Convergence

Pointwise convergence: Suppose $\{f_n\}$ is a sequence of functions defined on an interval I. The sequence $\{f_n\}$ converges pointwise to the limit function f on the interval I if $f_H(x) \approx f(x)$ for all standard $x \in I$ and for all infinitely large natural numbers H.

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The texts in red show the differences between pointwise and uniform convergence.

Sum Rule for Definite Integrals of Infinite Series

Our goal is to prove

$$\int_a^b \text{st}\left(\sum_{n=0}^{H_0} f_n(x)\right) dx = \text{st}\left(\sum_{n=0}^{H_1} \int_a^b f_n(x)dx\right) \tag{3}$$

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Our proof of (3) requires that a sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***continuous limit function*** on the interval of interest.

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Our proof of (3) requires that a sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***continuous limit function*** on the interval of interest. We come up with this requirement in two ways corresponding to two different conditions:

- Condition 1: A monotone sequence of partial sums of real-valued continuous functions ***converges pointwise*** to a ***continuous limit function*** on the closed and bounded interval of interest.
- Condition 2: A sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***limit function*** on the interval of interest.

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Requirement: A sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***continuous limit function*** on the interval of interest.

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  $\Rightarrow$ By Dini Uniform Convergence Theorem [W. A. J. Luxemburg, 1971], the sequence also ***converges uniformly*** to the ***continuous limit function***.

Dini Uniform Convergence Theorem

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A key technique in our proof of Dini's theorem is to apply the overspill principle from non-standard analysis [R. Goldblatt, 1998].

Overspill Principle

Weak version: Let $P(n, x)$ be a classical predicate. Then

$$\forall x.((\forall^{st} n \in \mathbb{N}.P(n, x)) \Rightarrow \exists^{\neg st} k \in \mathbb{N}.P(k, x))$$

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Strong version: Let $P(n, x)$ be a classical predicate. Then

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In words: If a classical predicate P holds for all standard natural numbers n, there must exist some non-standard natural number k s.t. P holds for all natural numbers less than or equal to k.

Sum Rule for Definite Integrals of Infinite Series

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  $\Rightarrow$ Using the overspill principle, we proved that the ***limit function*** is also ***continuous***.

Outline

Conclusions

We have extended a framework for formally evaluating definite integrals of real-valued continuous functions using FTC-2. Our framework can handle functions with free arguments.

We have formalized the Fourier coefficient formulas and the sum rule for definite integrals of infinite series in ACL2(r).

We have formalized the overspill principle in ACL2(r). We have built a simple interface that makes the overspill principle very easy to apply, thus strengthening the reasoning capability of non-standard analysis in ACL2(r). Our proofs of Dini's theorem and the continuity of the limit function illustrate this capability.

We are confident that our frameworks can be applied to future work on Fourier series and, more generally, continuous mathematics, to be carried out in ACL2(r).

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Thank You!