Computing Maximum Unavoidable Subgraphs Using SAT Solvers

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Outline

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Introduction

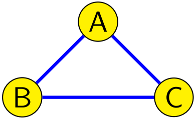
Definition 1 (Graph isomorphism)

Two graphs G and H are isomorphic if there exists an edge-preserving bijection from the vertices of G to the vertices of H. Isomorphic graphs occur in the same isomorphism class.

Definition 2 (Unavoidable subgraph)

A graph G is called an unavoidable subgraph (USG) of the **fully-connected graph** K_n if every red/blue edge-coloring of K_n contains an isomorphic graph of G **in only one color**.

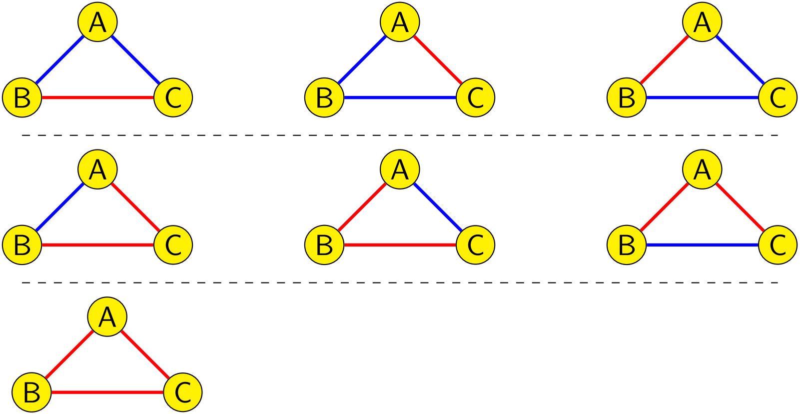
Introduction



For all red/blue edge-colorings of K_3,

exists a monochromatic path of two edges.

(Unavoidable subgraph)



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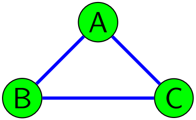
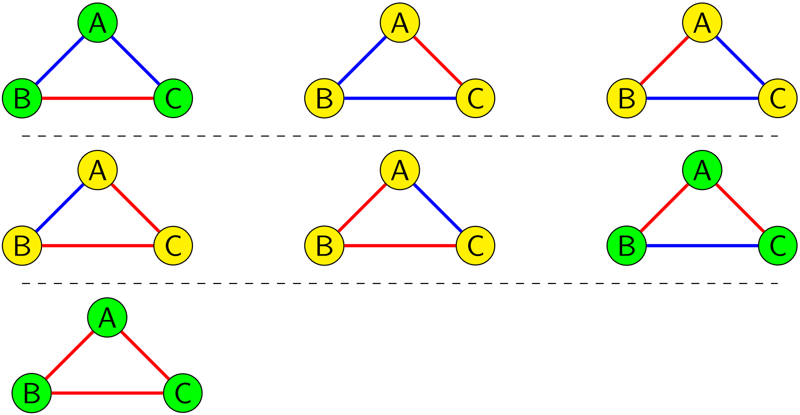


There exists a graph in each isomorphism class of K_3 s.t. the path B-A-C is monochromatic.



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The larger the USG (measured in the number of edges), the stronger the symmetry-breaking predicate (SBP) can be derived (explained in Section 4).

Outline

SAT Encoding of USGs

We employ a SAT solver to check whether a given graph G of order k is a USG of K_n ($k \leq n$).

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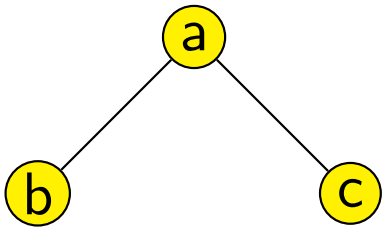
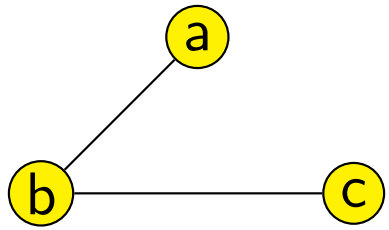
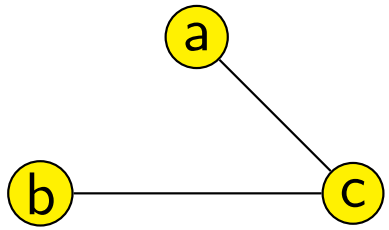
Encoding: Let's see how we encode the USG problem into SAT through the following example: *Check if a path of two edges is a USG of* K_3.







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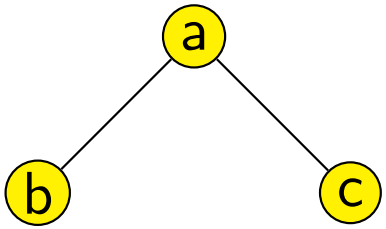

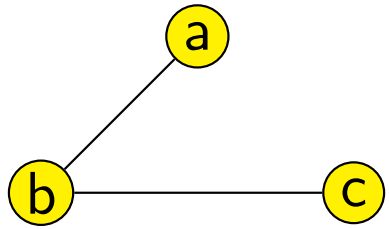

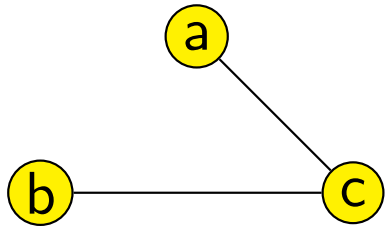

Let ***ab***, ***ac***, and ***bc*** denote the **Boolean variables representing the color of the edge** connecting vertices a and b, a and c, and b and c, respectively. If a Boolean variable has value T, the corresponding edge has color red. Otherwise it has color blue.

$$\mathcal{F} = (ab \wedge ac) \vee (\overline{ab} \wedge \overline{ac}) \vee (ab \wedge bc) \vee (\overline{ab} \wedge \overline{bc}) \vee (ac \wedge bc) \vee (\overline{ac} \wedge \overline{bc})$$

A path of two edges is a USG of K_3.
$\Leftrightarrow \mathcal{F}$ is VALID.
$\Leftrightarrow \overline{\mathcal{F}}$ is UNSATISFIABLE.

$$\overline{\mathcal{F}} = (\overline{ab} \vee \overline{ac}) \wedge (ab \vee ac) \wedge (\overline{ab} \vee \overline{bc}) \wedge (ab \vee bc) \wedge (\overline{ac} \vee \overline{bc}) \wedge (ac \vee bc)$$

Since $\overline{\mathcal{F}}$ is in CNF, SAT solvers can solve it directly.

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A graph G is unavoidable in $K_n \Leftrightarrow$ UNSAT$(\overline{\mathcal{F}_{G,K_n}} \wedge SBP(\overline{\mathcal{F}_{G,K_n}}))$.

Reducing the Maximum USG Search Space

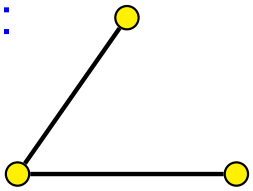
We detect the lower bound on the size and upper bound on the maximum degree of maximum USGs.

$\Rightarrow$ Using these bounds to reduce the maximum USG search space.

n	3	4	5	6	7	8	9
# isomorphism classes	4	11	34	156	1,044	12,346	274,668
# checked graphs	2	2	6	35	97	291	904

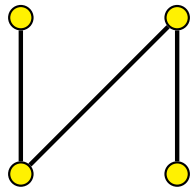
USG Results

K_3, K_4:



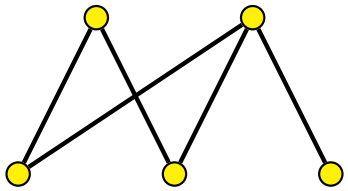
$|E| = 2$

K_5:



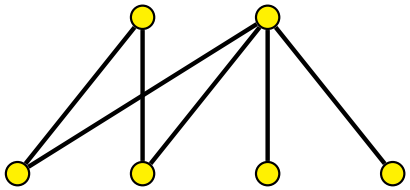
$|E| = 3$

K_6:



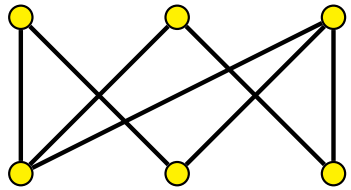
$|E| = 5$

K_7:



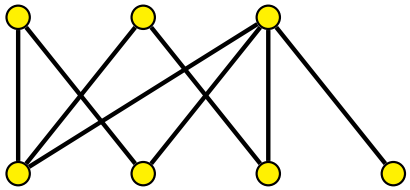
$|E| = 6$

K_8:



$|E| = 7$

K_9:



$|E| = 8$

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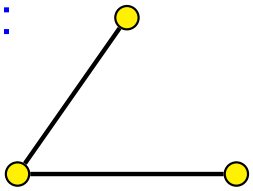
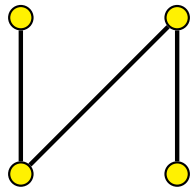


$|E| = 2$

K_5:



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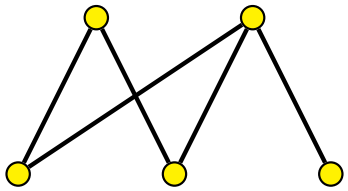

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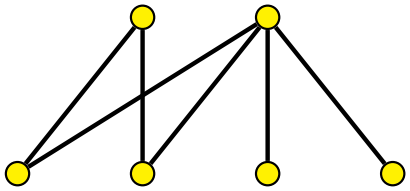

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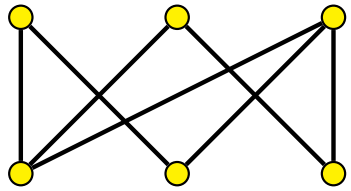

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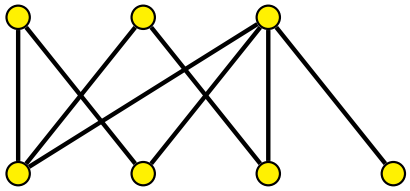

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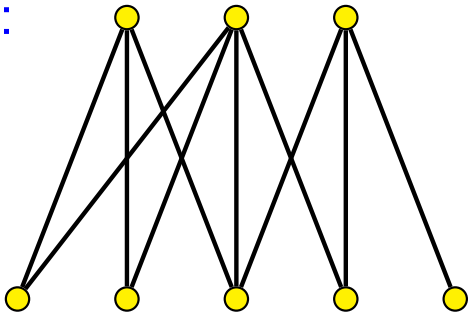
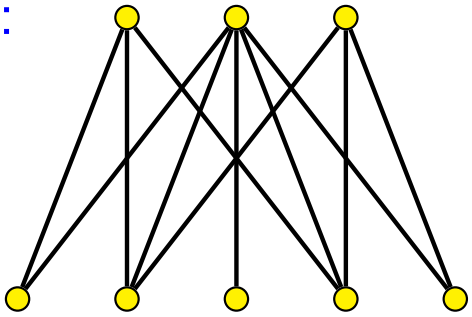
K_{10}:



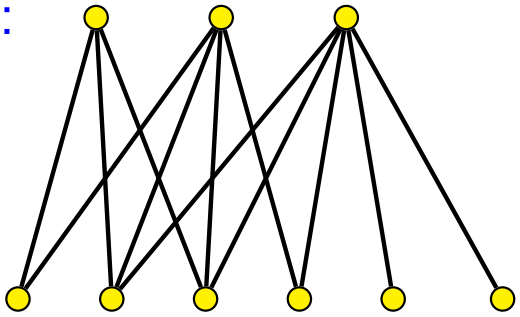
$|E| = 10$

K_{11}:



$|E| = 11$

K_{12}:



$|E| = 12$

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Multi-Component USG

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As heuristic to reduce the vast number of possible multi-component graphs, we restricted the search to graphs that have at most **three** components.

Multi-Component USG Results

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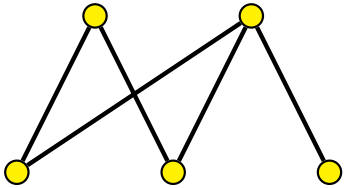

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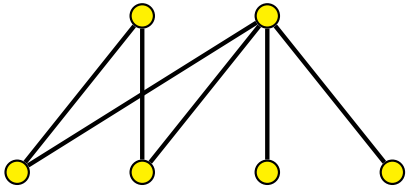

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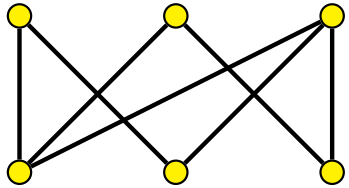

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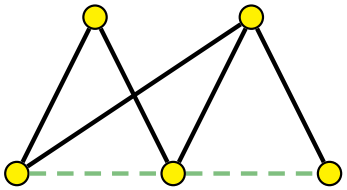

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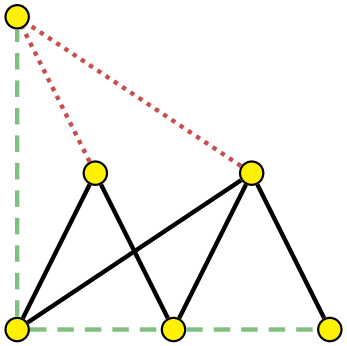

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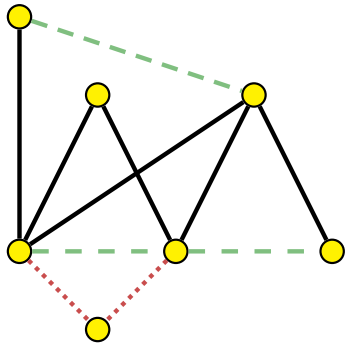

$|E| = 11$

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Deriving SBPs from USGs

Converting a USG G of K_n into an SBP by **forcing all edges in each component of G to be either all present or all absent**.

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Let e_1, e_2,..., e_ℓ denote the Boolean variables representing the edges of one component in G.

$$e_1 \leftrightarrow e_2 \leftrightarrow e_3 \leftrightarrow \cdots \leftrightarrow e_\ell \quad \equiv \quad e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow \cdots \rightarrow e_\ell$$

$$\equiv \quad (\overline{e_1} \vee e_2) \wedge (\overline{e_2} \vee e_3) \wedge \ldots \wedge (\overline{e_\ell} \vee e_1)$$

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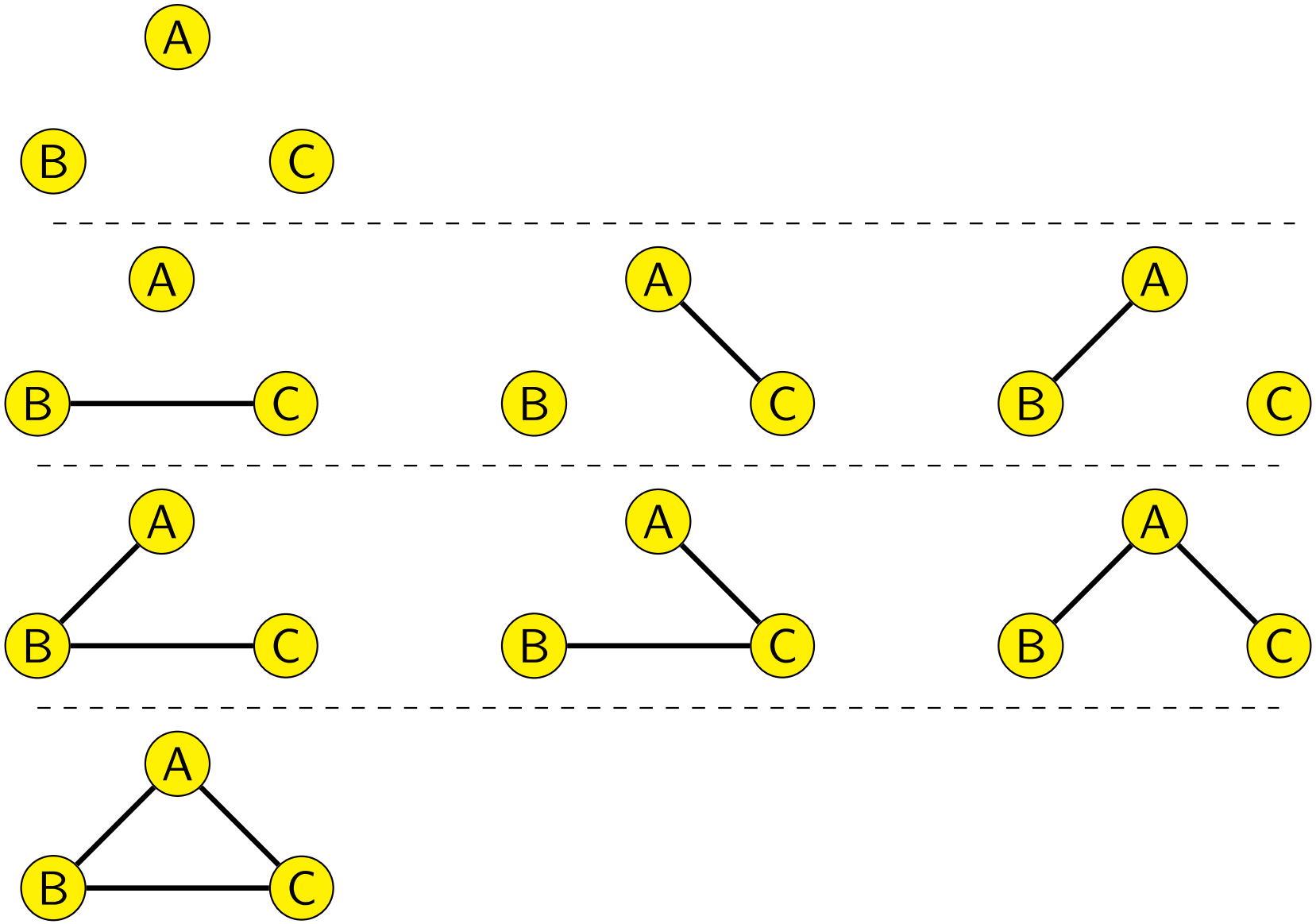
Applying this encoding for all components of G results in an SBP of $|E(G)|$ binary clauses.

Example



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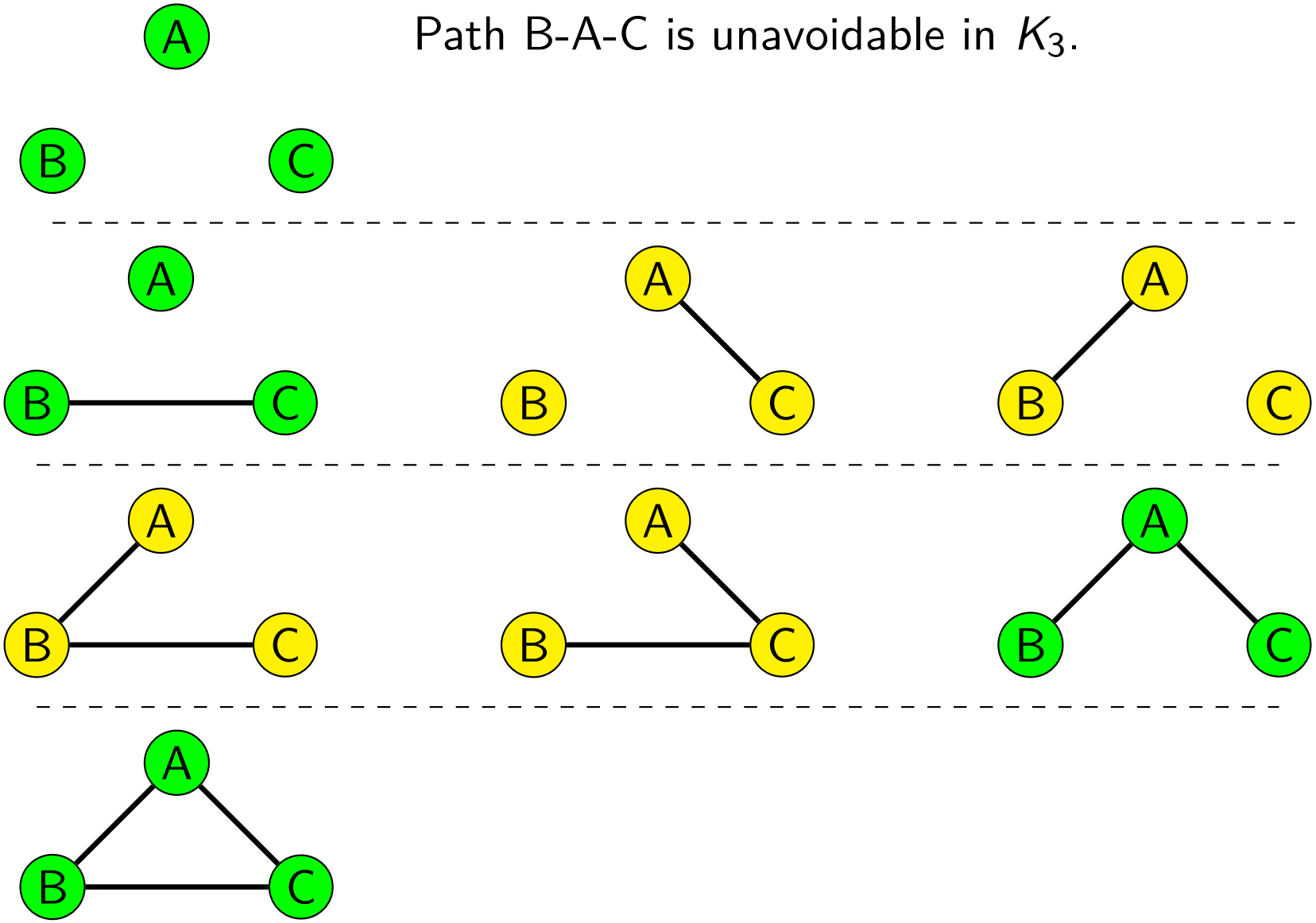

How Useful Are the USG-Based SBPs?

The number of graphs of order n is $2^{|E_{K_n}|}$, where

$$|E_{K_n}| = \binom{n}{2} = \frac{n(n-1)}{2}$$

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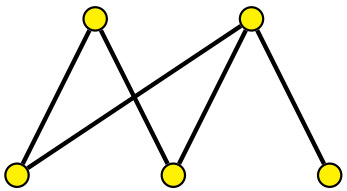
$\Rightarrow$ The larger the USG (measured in the number of edges), the stronger the SBP can be derived.

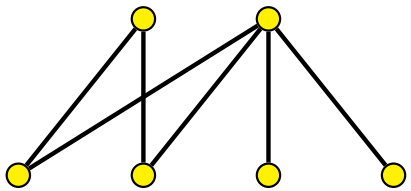
Single vs. Multi-Component USGs

K_6: $2^{|E(K_6)|} = 2^{15}$



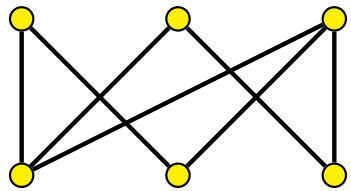
$2^{|E(K_6)|-5+1} = 2^{11}$

K_7: $2^{|E(K_7)|} = 2^{21}$

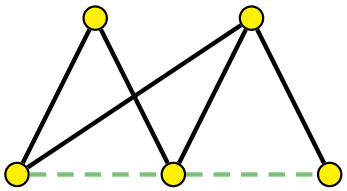


$2^{|E(K_7)|-6+1} = 2^{16}$

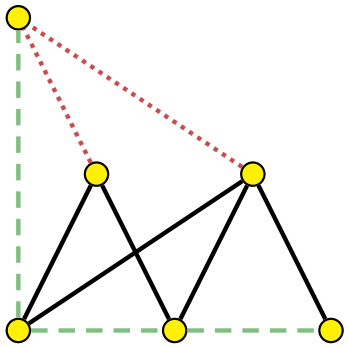
K_8: $2^{|E(K_8)|} = 2^{28}$



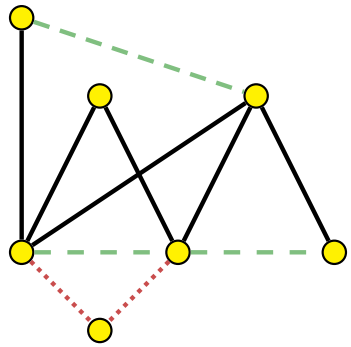
$2^{|E(K_8)|-7+1} = 2^{22}$



$2^{|E(K_6)|-7+2} = 2^{10}$



$2^{|E(K_7)|-10+3} = 2^{14}$



$2^{|E(K_8)|-11+3} = 2^{20}$

Runtime Results

Table: Runtime (in seconds) to compute the maximum or largest found single-component USGs of K_6 to K_{12} using the `glucose 3.0` SAT solver [G. Audemard and L. Simon, 2009]. The experiments were run on 3.5GHz Intel Xeon E31280 processors with 8MB L3 cache size. A '-' means a timeout after 24 hours.

n	6	7	8	9	10	11	12
No SBP	0	0.025	0.38	4.85	11,690.70	-	-
USG-SBP	0	0.002	**0.01**	**0.07**	26.45	266.94	532.48
L-SBP	0	**0**	**0.01**	0.11	**4.77**	**18.73**	312.60
Q-SBP	0	**0**	**0.01**	0.11	7.98	19.40	**303.40**

USG-SBP: SBP derived from the largest found single-component USG of K_{n-1}.

L-SBP: Linear SBP [F. Aloul et al., 2003].

Q-SBP: Quadratic SBP [M. Codish et al., 2013].

Outline

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We envision that knowledge about the maximum USGs, both the single and multi-component variants, could be a basis for novel symmetry-breaking techniques for SAT solvers.

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Questions?