Hidden Markov Models

Adapted from

Dr Catherine Sweeney-Reed's slides

Summary

- Introduction
- Description
- Central problems in HMM modelling
- Extensions
- Demonstration

Specification of an HMM

- *N - number of states*
  - $Q = \{q_1; q_2; : : : ;q_T\}$ *- set of states*
- *M - the number of symbols (observables)*
  - $O = \{o_1; o_2; : : : ;o_T\}$ *- set of symbols*

Specification of an HMM

- A - *the state transition probability matrix*
  - $aij = P(q_{t+1} = j|q_t = i)$
- B- *observation probability distribution*
  - $b_j(k) = P(o_t = k|q_t = j) \quad i \leq k \leq M$
- π - *the initial state distribution*

Specification of an HMM

- Full HMM is thus specified as a triplet:
  - $\lambda = (A,B,\pi)$

Central problems in HMM modelling

- <u>Problem 1</u>

  Evaluation:
  - Probability of occurrence of a particular observation sequence, $O = \{o_1,\ldots,o_k\}$, given the model
  - $P(O|\lambda)$
  - Complicated – hidden states
  - Useful in sequence classification

Central problems in HMM modelling

- Problem 2

  Decoding:
  - Optimal state sequence to produce given observations, $O = \{o_1, \ldots, o_k\}$, given model
  - Optimality criterion
  - Useful in recognition problems

Central problems in HMM modelling

- Problem 3

  Learning:
  - Determine optimum model, given a training set of observations
  - Find λ, such that P(O|λ) is maximal

Problem 1: Naïve solution

Central problems

- State sequence Q = (q_1,…q_T)
- Assume independent observations:

$$P(O \mid q, \lambda) = \prod_{i=1}^{T} P(o_t \mid q_t, \lambda) = b_{q1}(o_1) b_{q2}(o_2) \ldots b_{qT}(o_T)$$

NB Observations are mutually independent, given the hidden states. (Joint distribution of independent variables factorises into marginal distributions of the independent variables.)

Problem 1: Naïve solution

- Observe that :

$$P(q \mid \lambda) = \pi_{q1} a_{q1q2} a_{q2q3} \dots a_{qT-1qT}$$

- And that:

$$P(O \mid \lambda) = \sum_{q} P(O \mid q, \lambda) P(q \mid \lambda)$$

Problem 1: Naïve solution

- Finally get:

$$P(O \mid \lambda) = \sum_{q} P(O \mid q, \lambda) P(q \mid \lambda)$$

NB:
-The above sum is over all state paths
-There are N^T states paths, each 'costing' $O(T)$ calculations, leading to $O(TN^T)$ time complexity.

Problem 1: Efficient solution

Forward algorithm:

- Define auxiliary forward variable α:

$$\alpha_t(i) = P(o_1, \ldots, o_t \mid q_t = i, \lambda)$$

$\alpha_t(i)$ is the probability of observing a partial sequence of observables $o_1, \ldots o_t$ such that at time t, state $q_t = i$

Problem 1: Efficient solution

- Recursive algorithm:
  - Initialise:

$$\alpha_1(i) = \pi_i b_i(o_1)$$

  - Calculate:

(Partial obs seq to *t* AND state *i* at *t*) x (transition to *j* at *t+1*) x (sensor)

$$\alpha_{t+1}(j) = [\sum_{i=1}^{N} \alpha_t(i) a_{ij}] b_j(o_{t+1})$$

Sum, as can reach *j* from any preceding state

α incorporates partial obs seq to *t*

  - Obtain:

$$P(O \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

Sum of different ways of getting obs seq

Complexity is $O(N^2T)$

Problem 1: Alternative solution

Backward algorithm:

- Define auxiliary forward variable *β*:

$$\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T \mid q_t = i, \lambda)$$

$\beta_t(i)$ – the probability of observing a sequence of observables $o_{t+1}, \ldots, o_T$ given state $q_t = i$ at time *t,* and λ

Problem 1: Alternative solution

- Recursive algorithm:
  - Initialise:

$$\beta_T(j) = 1$$

  - Calculate:

$$\beta_t(i) = \sum_{j=1}^{N} \beta_{t+1}(j) a_{ij} b_j(o_{t+1})$$

  - Terminate:

$$p(O \mid \lambda) = \sum_{i=1}^{N} \beta_1(i) \qquad t = T-1,\ldots,1$$

Complexity is $O(N^2T)$

Problem 2: Decoding

- Choose state sequence to maximise probability of observation sequence
- Viterbi algorithm - inductive algorithm that keeps the best state sequence at each instance

Problem 2: Decoding

Viterbi algorithm:

- State sequence to maximise P(O,Q|λ):

$$P(q_1, q_2, \ldots q_T \mid O, \lambda)$$

- Define auxiliary variable δ:

$$\delta_t(i) = \max_q P(q_1, q_2, \ldots, q_t = i, o_1, o_2, \ldots o_t \mid \lambda)$$

$\delta_t(i)$ – the probability of the most probable path ending in state q_t=i

Central problems

Problem 2: Decoding

To get state seq, need to keep track of argument to maximise this, for each *t* and *j*. Done via the array $\psi_t(j)$.

- Recurrent property:

$$\delta_{t+1}(j) = \max_i(\delta_t(i)a_{ij})b_j(o_{t+1})$$

- Algorithm:
  - 1. Initialise:

$$\delta_1(i) = \pi_i b_i(o_1) \qquad 1 \le i \le N$$

$$\psi_1(i) = 0$$

Problem 2: Decoding

Central problems

- 2. Recursion:

$$\delta_t(j) = \max_{1 \le i \le N}(\delta_{t-1}(i)a_{ij})b_j(o_t)$$

$$\psi_t(j) = \arg\max_{1 \le i \le N}(\delta_{t-1}(i)a_{ij}) \qquad 2 \le t \le T, 1 \le j \le N$$

- 3. Terminate:

P* gives the state-optimised probability

$$P^* = \max_{1 \le i \le N} \delta_T(i)$$

$$q_T^* = \arg\max_{1 \le i \le N} \delta_T(i)$$

Q* is the optimal state sequence (Q* = {q1*,q2*,…,qT*})

Problem 2: Decoding

Central problems

- 4. Backtrack state sequence:

$$q_t^* = \psi_{t+1}(q_{t+1}^*) \qquad t + T - 1, T - 2, \ldots, 1$$

O(N^2T) time complexity

Problem 3: Learning

- Training HMM to encode obs seq such that HMM should identify a similar obs seq in future
- Find $\lambda=(A,B,\pi)$, maximising $P(O|\lambda)$
- General algorithm:
  - Initialise: λ_0
  - Compute new model λ, using λ_0 and observed sequence O
  - Then $\lambda_o \leftarrow \lambda$
  - Repeat steps 2 and 3 until:

$$\log P(O \mid \lambda) - \log P(O \mid \lambda_0) < d$$

Problem 3: Learning

Step 1 of Baum-Welch algorithm:

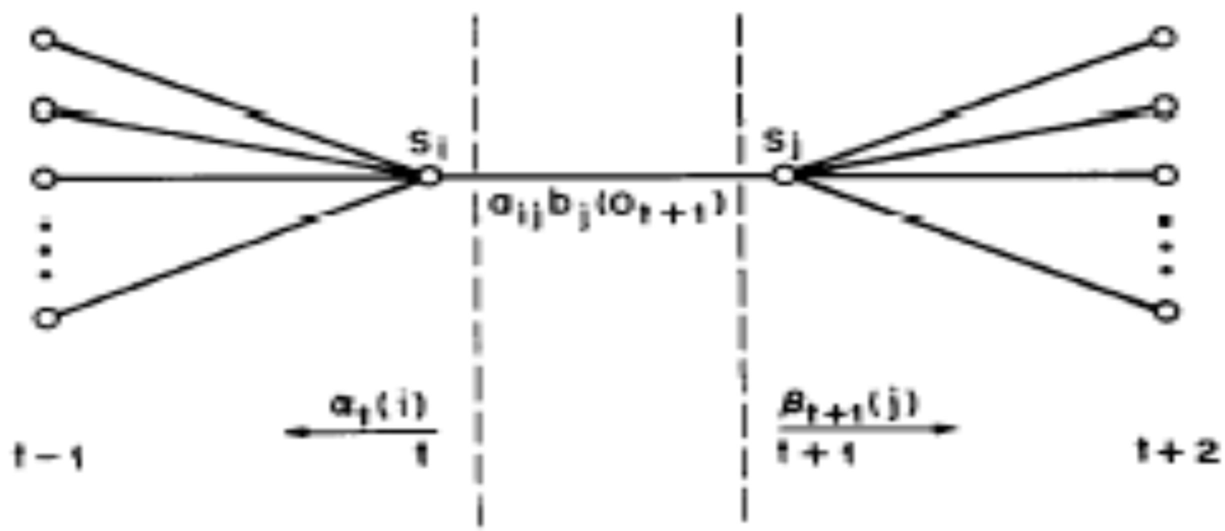
- Let *ξ(i,j)* be a probability of being in state *i* at time *t* and at state *j* at time *t+1*, given λ and O seq

$$\xi(i,j) = \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{P(O \mid \lambda)}$$

$$= \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}$$

Central problems

Problem 3: Learning



Operations required for the computation of the joint event that the system is in state Si and time t and State Sj at time t+1

Problem 3: Learning

Central problems

- Let $\gamma_t(i)$ be a probability of being in state *i* at time *t*, given *O*

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i, j)$$

- $\sum_{t=1}^{T-1} \gamma_t(i)$ - expected no. of transitions from state *i*
- $\sum_{t=1}^{T-1} \xi_t(i)$ - expected no. of transitions $i \rightarrow j$

Problem 3: Learning

Central problems

Step 2 of Baum-Welch algorithm:

- $\hat{\pi} = \gamma_1(i)$ the expected frequency of state *i* at time *t=1*
- $\hat{a}_{ij} = \frac{\sum \xi_t(i,j)}{\sum \gamma_t(i)}$ ratio of expected no. of transitions from state *i* to *j* over expected no. of transitions from state *i*
- $\hat{b}_j(k) = \frac{\sum_{t,o_t=k} \gamma_t(j)}{\sum \gamma_t(j)}$ ratio of expected no. of times in state *j* observing symbol *k* over expected no. of times in state *j*

Problem 3: Learning

- Baum-Welch algorithm uses the forward and backward algorithms to calculate the auxiliary variables α,β
- B-W algorithm is a special case of the EM algorithm:
  - E-step: calculation of ξ and γ
  - M-step: iterative calculation of $\hat{\pi}$, $\hat{a}_{ij}$, $\hat{b}_j(k)$
- Practical issues:
  - Can get stuck in local maxima
  - Numerical problems – log and scaling

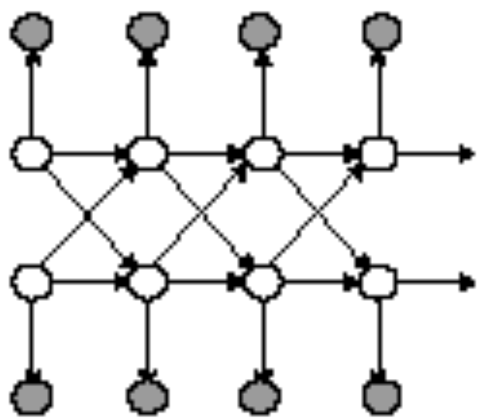
Extensions

- Problem-specific:
  - Left to right HMM (speech recognition)
  - Profile HMM (bioinformatics)

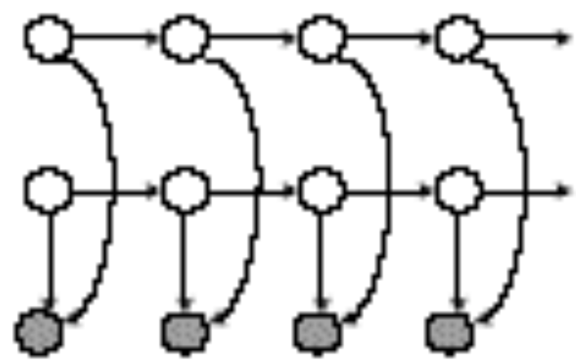
Extensions

- General machine learning:
  - Factorial HMM
  - Coupled HMM
  - Hierarchical HMM
  - Input-output HMM
  - Switching state systems
  - Hybrid HMM (HMM +NN)
  - Special case of graphical models
    - Bayesian nets
    - Dynamical Bayesian nets

Examples



Coupled HMM

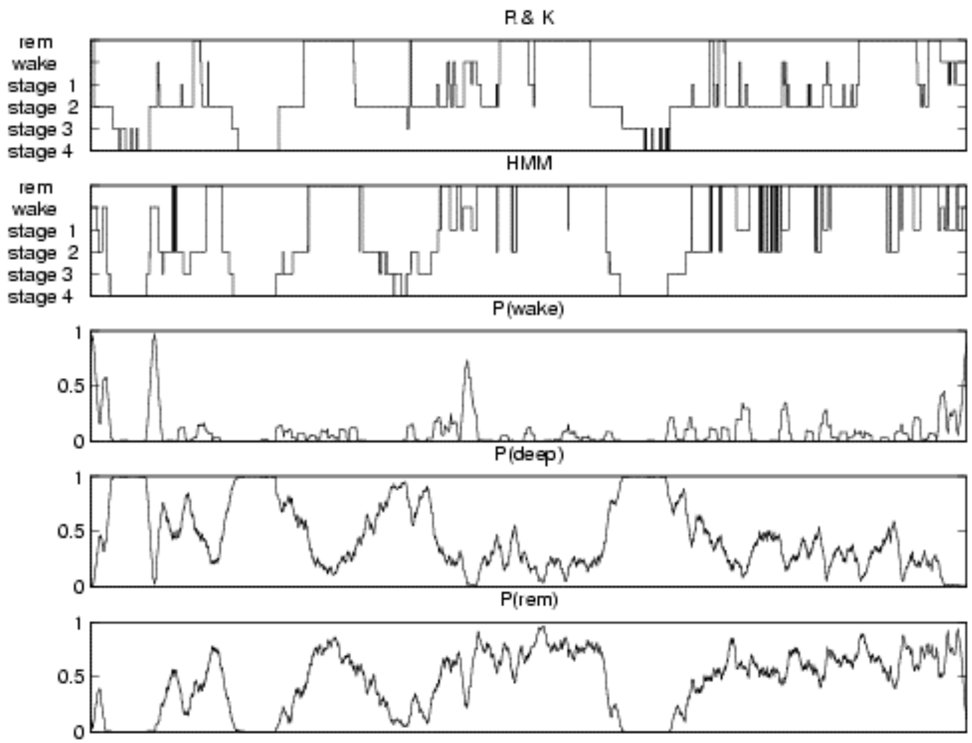


Factorial HMM

HMMs – Sleep Staging

- Flexer, Sykacek, Rezek, and Dorffner (2000)
- Observation sequence: EEG data
- Fit model to data according to 3 sleep stages to produce continuous probabilities: P(wake), P(deep), and P(REM)
- Hidden states correspond with recognised sleep stages. 3 continuous probability plots, giving P of each at every second

HMMs – Sleep Staging



Manual scoring of sleep stages

Staging by HMM

Probability plots for the 3 stages

Excel

- Demonstration of a working HMM implemented in Excel

Further Reading

- L. R. Rabiner, "A tutorial on Hidden Markov Models and selected applications in speech recognition," *Proceedings of the IEEE*, vol. 77, pp. 257-286, 1989.
- R. Dugad and U. B. Desai, "A tutorial on Hidden Markov models," Signal Processing and Artifical Neural Networks Laboratory, Dept of Electrical Engineering, Indian Institute of Technology, Bombay Technical Report No.: SPANN-96.1, 1996.
- W.H. Laverty, M.J. Miket, and I.W. Kelly, "Simulation of Hidden Markov Models with EXCEL", The Statistician, vol. 51, Part 1, pp. 31-40, 2002