Eigenvalues for high dim. data

$$\Sigma = \frac{1}{M} \sum_{n=1}^{M} \boldsymbol{X}_n \boldsymbol{X}_n^T$$

= 1/M AA^T

where

$$A = [\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_M]$$

an $M \times N$ matrix of M data samples.

Rather than finding the eigenvectors of the larger system, consider finding the eigenvectors of the $M \times M$ system

$$A^T A \boldsymbol{v} = \mu \boldsymbol{v} \tag{1}$$

Premultiplying both sides by A,

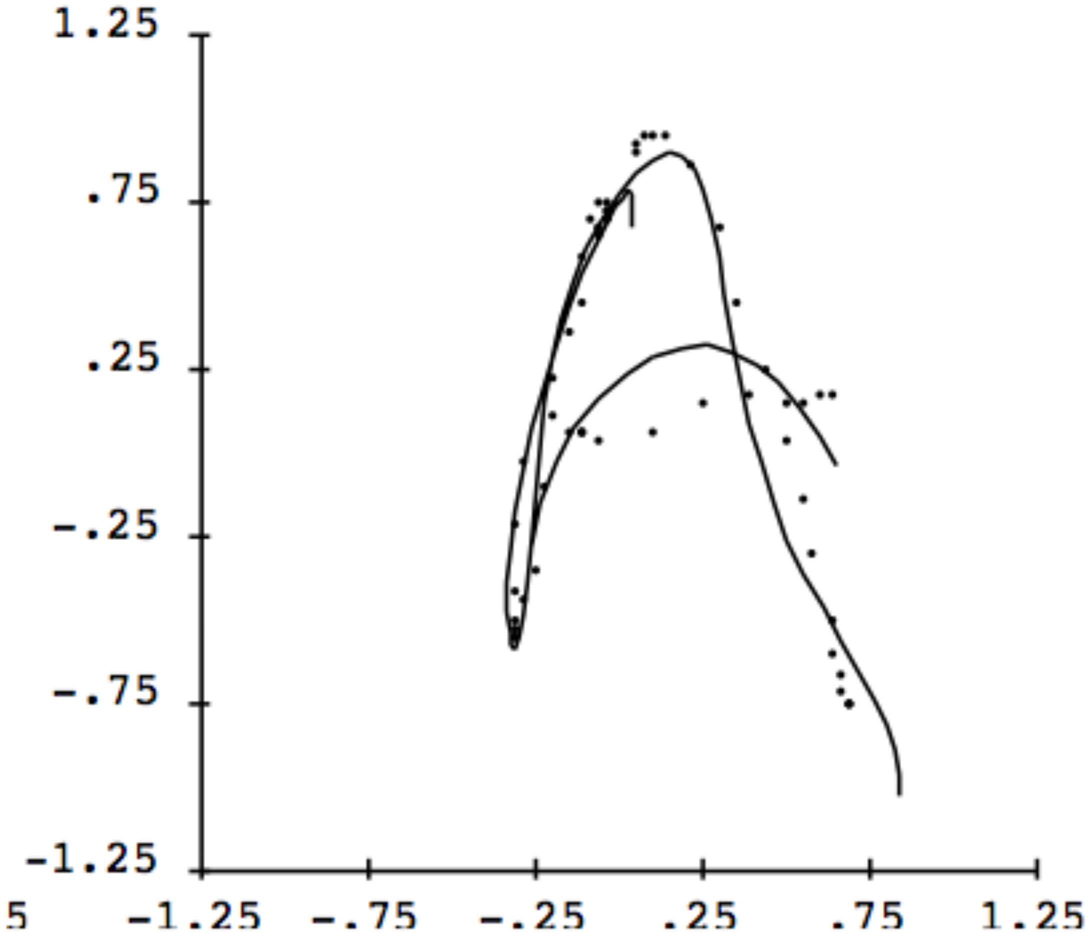
$$AA^T A \boldsymbol{v} = \mu A \boldsymbol{v}$$

MACHINE LEARNING

WEEK 2: LINEAR ALGEBRA

LECTURE 2: DYNAMICS

Example Generating the letter ‘A’ w dynamics



Cf digit recognition strategy

Dynamical Systems basics

Dynamical systems can be summarized as a single vector equation

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$$

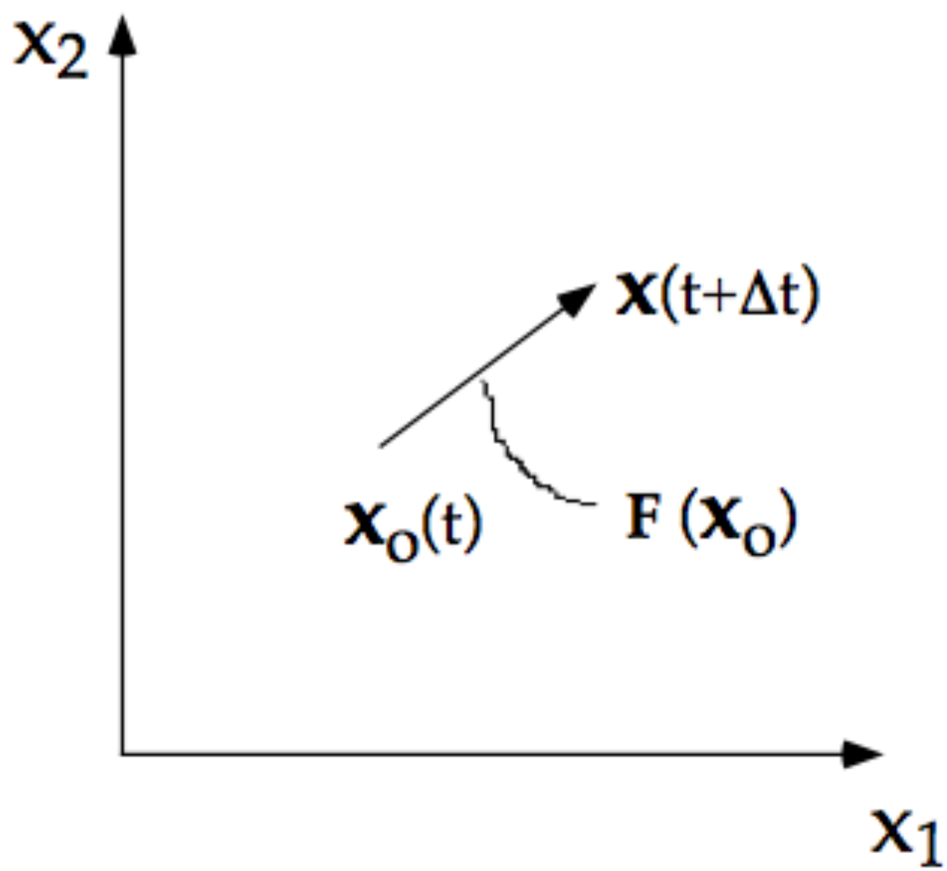
where $\dot{\boldsymbol{x}}$, read as "x dot," denotes temporal derivative $\frac{d\boldsymbol{x}}{dt}$ and, in vector notation,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{and } \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}$$

and

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix} \tag{1}$$

Interpretation of F



Linearizing a NL System

Example: Linearizing a Dynamical System For a second-order system let

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} x_1 + x_2^2 \\ x_1 x_2 + x_2^2 \end{pmatrix}$$

Then the Jacobian is given by

$$\boldsymbol{F}'(\boldsymbol{x}) = \begin{bmatrix} 1 & 2x_2 \\ x_2 & x_1 + 2x_2 \end{bmatrix}$$

At equilibrium, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$, so that the point $\boldsymbol{x}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an equilibrium point. At this point the Jacobian $\boldsymbol{F}'(\boldsymbol{x}_0)$ simplifies to

$$\boldsymbol{F}'(\boldsymbol{x}_0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Example

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General Case

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

where A is a matrix of constant coefficients.

How do we solve this system? Since $x = x_0 e^{at}$ worked before, try

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 e^{At} \tag{4}$$

where $\boldsymbol{x}_0$ is a vector of initial conditions. What about e^{At}? Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

define e^A as

$$e^A \equiv I + A + \frac{A^2}{2} + \cdots$$

and substitute into both sides of $\dot{\boldsymbol{x}} = A\boldsymbol{x}$:

$$\begin{aligned} A\boldsymbol{x} &= A[I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots]\boldsymbol{x}_0 \\ \dot{\boldsymbol{x}} &= [A + \frac{2A(At)}{2!} + \frac{3A(At)^2}{3!} + \cdots]\boldsymbol{x}_0 \end{aligned}$$

Suppose A is diagonal

$$A = \Lambda = \begin{bmatrix} \lambda_1 & & 0 & \\ & \lambda_2 & & \\ & & \ddots & \\ & 0 & & \lambda_n \end{bmatrix}$$

In this case, all the equations are decoupled and

$$e^A = e^\wedge = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}$$

so that solving this system reduces to solving n versions of the one-dimensional case. That is,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = e^{\wedge t}\boldsymbol{x}(0) = \begin{bmatrix} x_1(0)e^{\lambda_1 t} \\ x_2(0)e^{\lambda_2 t} \\ \vdots \\ x_n(0)e^{\lambda_n t} \end{bmatrix}$$

Similarity

$$\boldsymbol{x}^* = A\boldsymbol{x}$$

$$\boldsymbol{y}^* = A\boldsymbol{y}$$

Given the transformation

$$\boldsymbol{y} = W\boldsymbol{x}$$

what happens to W when the coordinate system is changed to the starred system? That is, for some W^* it will be true that

$$\boldsymbol{y}^* = W^*\boldsymbol{x}^*$$

What is the relation between W and W^*? One way to find out is to change back to the original system, transform by W, and then transform back to the starred system; that is,

$$\boldsymbol{x} = A^{-1}\boldsymbol{x}^*$$

$$\boldsymbol{y} = W\boldsymbol{x}$$

$$\boldsymbol{y}^* = A\boldsymbol{y}$$

Putting these transformations together:

$$\boldsymbol{y}^* = AWA^{-1}\boldsymbol{x}^*$$

Since the vector transformation taken by the two different routes should be the same, it must be true that

$$W^* = AWA^{-1}$$

How to diagonalize a sq. matrix

Now let's relate this discussion to eigenvectors. Suppose that the eigenvectors have been chosen as the basis set. Then for a given eigenvector $\boldsymbol{y}_i$,

$$W\boldsymbol{y}_i = \lambda \boldsymbol{y}_i$$

and if Y is a matrix whose columns are the eigenvectors $\boldsymbol{y}_i$, then

$$WY = Y\Lambda$$

Here Λ is a matrix whose only nonzero components are the diagonal elements λ_i. Premultiplying both sides by Y^{-1},

$$Y^{-1}WY = \Lambda$$

Meaning of eigenvalues in dx/dt=Ax

There are 2 types of stable equilibria in a two-dimensional space: **knot** and **focus**

$\lambda_1 = -0.5$
$\lambda_2 = -1.3$
Knot: eigenvalues have no imaginary part

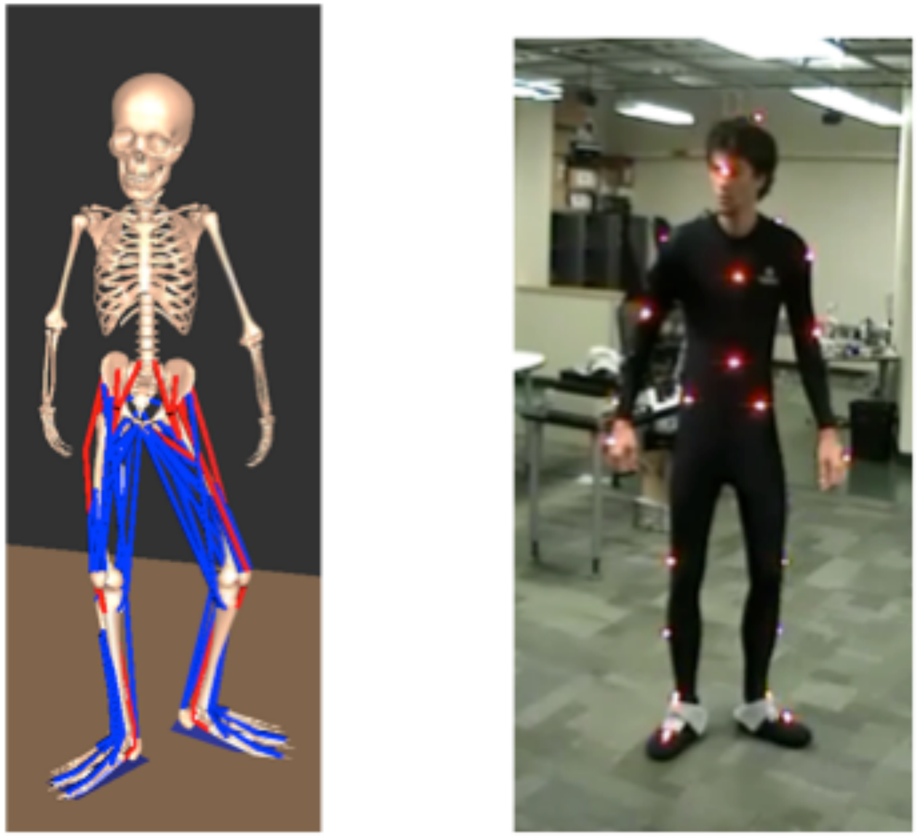
$\lambda_1 = -0.8 - 1.5i$
$\lambda_2 = -0.8 + 1.5i$
Focus: eigenvalues have imaginary parts

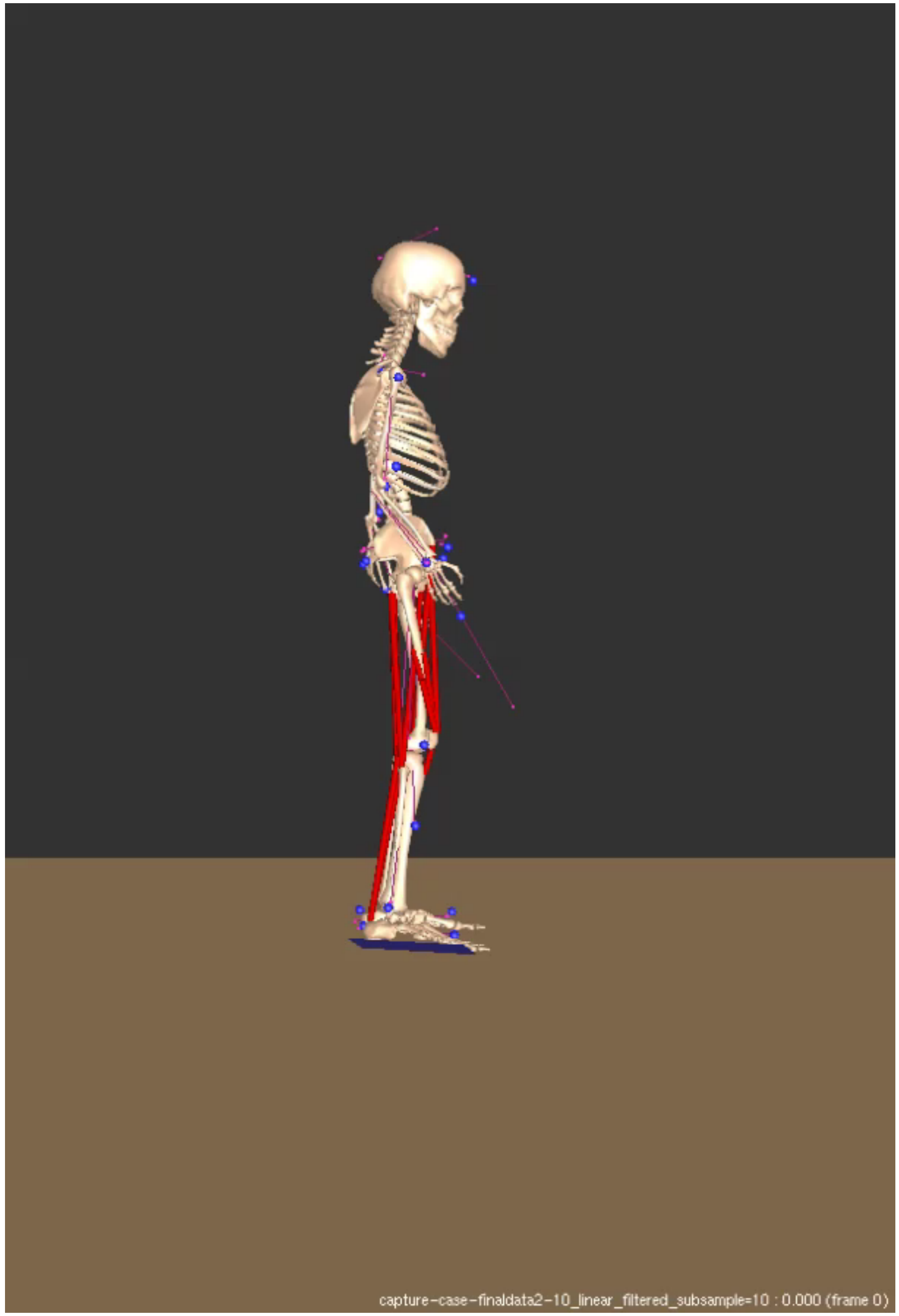
There are 3 types of unstable equilibria in a two-dimensional space: **knot**, **focus**, and **saddle**

Unstable *knot*
$\lambda_1 = 0.5$
$\lambda_2 = 0.3$

Unstable *focus*
$\lambda_1 = 0.5 - 0.2\,i$
$\lambda_2 = 0.5 + 0.2\,i$

Saddle
$\lambda_1 = 0.5$
$\lambda_2 = -0.3$

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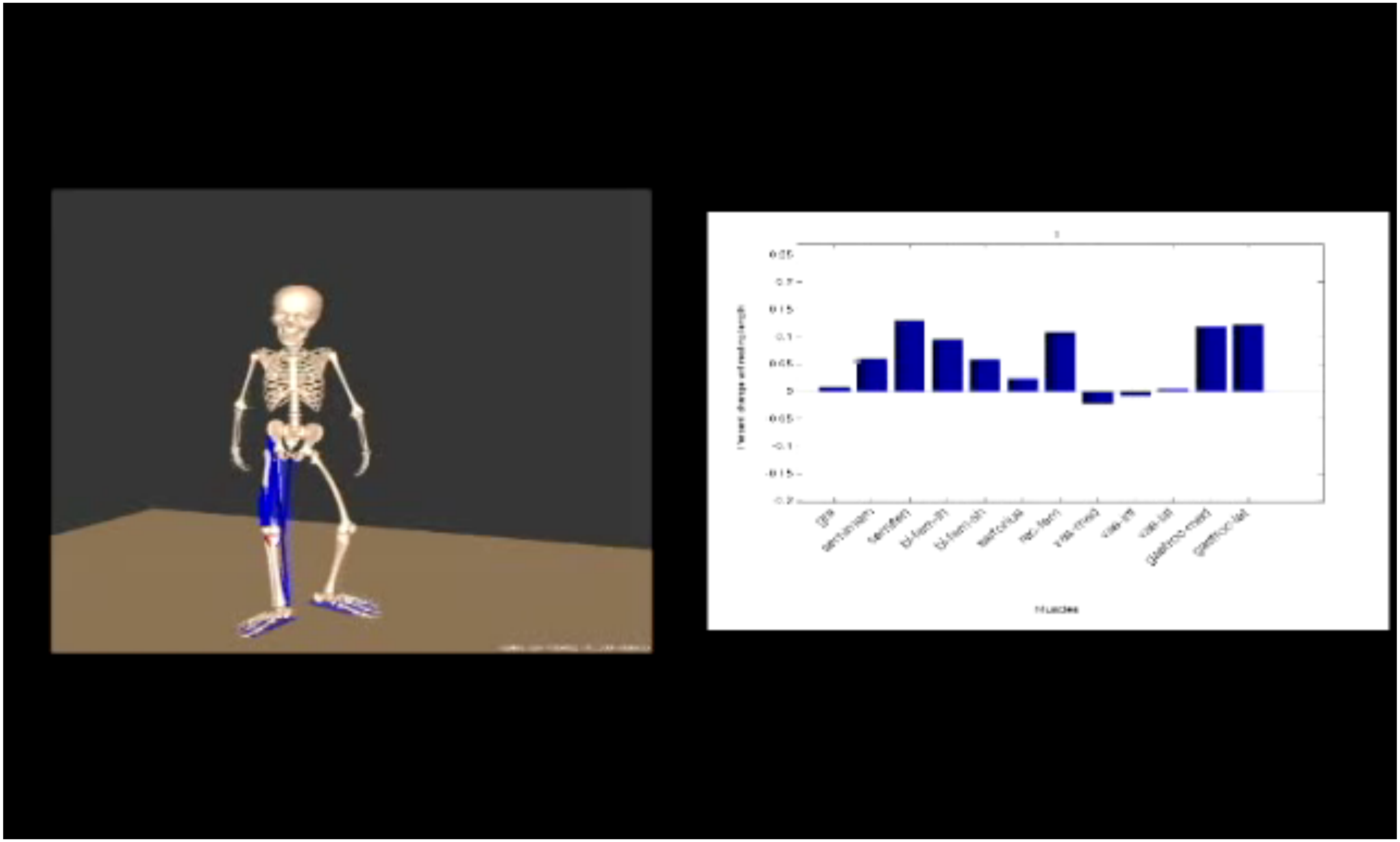

Iyer(2010) Neural Control of Momement

Iyer & Ballard (2009BioRob Conference

Reconstruction using Gabor-like basis functions