MACHINE LEARNING

WEEK 3: OPTIMIZATION

LECTURE 2: HAMILTONIAN, KALMAN FILTER
Euler-Lagrange Method

\[
\max_u J \text{ subject to the constraint } \dot{x} = F(x, u)
\]

The strategy will be to assume that \( u \) maximizes \( J \) and then use this assumption to derive other conditions for a maximum. These arguments depend on making a perturbation in \( u \) and seeing what happens. Since \( u \) affects \( x \), the calculations become a little involved, but the argument is just a matter of careful bookkeeping. The main trick is to add additional terms to \( J \) that sum to zero. Let’s start by appending the dynamic equation to \( J \) as before, but this time using continuous Lagrange multipliers \( \lambda(t) \):

\[
\bar{J} = J - \int_0^T \lambda^T [\dot{x} - F(x, u)] dt
\]
Anticipating what is about to happen, we define the Hamiltonian $H(\lambda, x, u)$ as

$$H(\lambda, x, u) \equiv \lambda^T [F(x, u)] + \ell(x, u)$$

so that the expression for $\bar{J}$ becomes

$$\bar{J} = \psi[x(T)] + \int_0^T [H(\lambda, x, u) - \lambda^T \dot{x}] dt$$
Now let's examine the effects of a small change in \( u \), as shown in Figure 6 on \( \bar{J} \), just keeping track of the change \( \delta \bar{J} \):

\[
\delta \bar{J} = \psi[x(T) + \delta x(T)] - \psi[x(T)] + \int_0^T [H(\lambda, x + \delta x, v) - H(\lambda, x, u) - \lambda^T \delta x] dt
\]
Integration by parts

Using the expression for integration by parts for $\int \lambda^T \delta \dot{x} dt$:

$$\int_0^T \lambda^T \delta \dot{x} \, dt = \lambda^T(T) \delta x(T) - \lambda^T(0) \delta x(0) - \int_0^T \lambda^T \delta x \, dt$$

Now substitute this into the expression for $\delta \bar{J}$,

$$\delta \bar{J} = \psi[x(T) + \delta x(T)] - \psi[x(T)] - \lambda(T)^T \delta x(T)$$
$$+ \int_0^T [H(\lambda, x + \delta x, u) - H(\lambda, x, u) + \lambda^T \delta x] \, dt$$
Variational analysis

Now concentrate just on the first two terms in the integral:

$$\int_0^T [H(\lambda, x + \delta x, u) - H(\lambda, x, u)] dt$$

First add and subtract $H(\lambda, x, u)$:

$$= \int_0^T [H(\lambda, x + \delta x, u) - H(\lambda, x, u) + H(\lambda, x, u) - H(\lambda, x, u)] dt$$

Next expand the first term inside the integral in a Taylor series and neglect terms above first order,

$$\approx \int_0^T (H_x(\lambda, x, u)^T \delta x + H(\lambda, x, u) - H(\lambda, x, u)) dt$$

where $H_x$ is the partial $\frac{\partial H}{\partial x}$, which is

$$\begin{pmatrix}
H_{x_1} \\
\vdots \\
H_{x_n}
\end{pmatrix}$$
Now add and subtract $H_x(\lambda, x, u)^T \delta x$:

$$= \int_0^T \{ H_x(\lambda, x, u)^T \delta x + [H_x(\lambda, x, v) - H_x(\lambda, x, u)]^T \delta x + H(\lambda, x, v) - H(\lambda, x, u) \} dt$$

The term $[H_x(\lambda, x, v) - H_x(\lambda, x, u)]^T \delta x$ can be neglected because it is the product of two small terms, $[H_x(\lambda, x, v) - H_x(\lambda, x, u)]$ and $\delta x$, and thus is a second-order term. Thus

$$\approx \int_0^T (H_x(\lambda, x, u) \delta x + H(\lambda, x, v) - H(\lambda, x, u)) dt$$
Finally, substitute this expression back into the original equation for $\delta J$, yielding

$$
\delta \bar{J} \approx \{ \psi_x[\dot{x}(T)] - \lambda^T(T) \} \delta x(T)
+ \int_0^T [H_x(\lambda, x, u) + \dot{\lambda}^T] \delta x \, dt
+ \int_0^T [H(\lambda, x, v) - H(\lambda, x, u)] dt
$$

Since we have the freedom to pick $\lambda$, just to make matters simpler, pick it so that the first integral vanishes:

$$
-\dot{\lambda}^T = H_x(\lambda, x, u)
\lambda^T(T) = \psi_x[x(T)]
$$
Condition for a maximum

Now all $\delta \bar{J}$ has left is

$$\delta \bar{J} = \int_0^T [H(\lambda, x, v) - H(\lambda, x, u)] dt$$

From this equation it follows that the optimal control $u^*$ must be such that

$$H(\lambda, x, u^*) \geq H(\lambda, x, u), \ u \in U \quad (3)$$

To see this point, suppose that it were not true, that is, that for some interval of time there was a $v \in U$ such that

$$H(\lambda, x, v) \geq H(\lambda, x, u^*)$$

This assumption would mean that you could adjust the integral so that the perturbation $\delta \bar{J}$ is positive, contradicting the original assumption that $\bar{J}$ is maximized by $u^*$. Therefore Equation 3 must hold.
Summary

In addition to the dynamic equations

$$\dot{x} = f(x, u)$$

and associated initial condition

$$x(0) = x_0$$

the Lagrange multipliers also must obey a constraint equation

$$-\dot{\lambda}^T = Hx$$

that has a final condition

$$\lambda^T(T) = \psi_x[x(T)]$$

The equation for $\lambda$ is known as the adjoint equation. In addition, for all $t$, the optimal control $u$ is such that

$$H[\lambda(t), x(t), u] \leq H[\lambda(t), x(t), u(t)]$$

where $H$ is the Hamiltonian

$$H = \lambda^T f(x, u) + \ell(x, u)$$
The dynamic equation is
\[ \ddot{x} = -\dot{x} + u(t) \]

with initial conditions
\[ x(0) = 0, \quad \dot{x}(0) = 0 \]

The cost functional
\[ J = x(T) - \frac{1}{2} \int_0^T u^2(t) dt \]
captures the desire to maximize the distance traveled in time \( T \) and at the same time penalize excessive accelerations.

Using the transformation of Section 5.2.1, the state variables \( x_1 \) and \( x_2 \) are defined by
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
x_2 \\
-x_2 + u
\end{pmatrix}
\]
\[ x_1(0) = x_2(0) = 0 \]
\[ J = x_1(T) - \frac{1}{2} \int_0^T u^2 dt \]

The Hamiltonian is given by
\[ H = \lambda_1 x_2 - \lambda_2 x_2 + \lambda_2 u - \frac{1}{2} u^2 \]
Differentiating this equation allows the determination of the adjoint system as

\[-\dot{\lambda}_1 = \frac{\partial H}{\partial x_1} = 0\]

\[-\dot{\lambda}_2 = \frac{\partial H}{\partial x_2} = \lambda_1 - \lambda_2\]

and its final condition can be determined from

\[\psi = x_1(T)\]

\[\lambda(T) = \psi x(T) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\]

The simple form of the adjoint equations allows their direct solution. For \(\lambda_1\),

\[\lambda_1 = \text{const} = 1\]

For \(\lambda_2\), we could use Laplace transform methods, but they have not been discussed, so let's make the incredible lucky guess:

\[\lambda_2 = 1 - e^{t-T}\]

For a maximum differentiate \(H\) with respect to \(u\),

\[\frac{\partial H}{\partial u} = 0 \Rightarrow \lambda_2 - u = 0\]

\[u = \lambda_2 = 1 - e^{t-T}\]
Kalman Filter

Suppose you are able to make measurements $z$ that are related to a variable that you would like to know by a linear relationship:

$$z = Hx + \nu$$

The term $\nu$ represents unwanted noise and is assumed to have the statistics

$$E(\nu) = 0$$
and

$$E(\nu^T \nu) = R$$

Somehow you have already estimated the variable $x$ as $\bar{x}$.

Now you would like to use $z$ to improve the estimate of $x$. A logical way to proceed would be to weight the two different sources of knowledge, $z$ and $\bar{x}$.

$$J(x) = \frac{1}{2}[(x-\bar{x})^T M^{-1}(x-\bar{x}) + (z-Hx)^T R^{-1}(z-Hx)]$$
Kalman Filter

Suppose you are able to make measurements $z$ that are realated to a variable that you would like to know by a linear relationship:

$$z = Hx + v$$

The term $v$ represents unwanted noise and is assumed to have the statistics

$$E(v) = 0$$

and

$$E(vv^T) = R$$

Somehow you have already estimated the variable $x$ as $\bar{x}$.

Now you would like to use $z$ to improve the estimate of $x$. A logical way to proceed would be to weight the two different sources of knowledge, $z$ and $\bar{x}$.

$$J(x) = \frac{1}{2}[(x-\bar{x})^T M^{-1}(x-\bar{x}) + (z-Hx)^T R^{-1}(z-Hx)]$$
At the minimum,

\[ dJ = 0 = dx^T [M^{-1}(x - \bar{x}) - H^T R^{-1}(z - Hx)] \]

To satisfy this in general, the term in [] must be zero

\[ M^{-1}(\hat{x} - \bar{x}) - H^T R^{-1}(z - H\hat{x}) = 0 \]

\[ (M^{-1} - H^T R^{-1}H)\hat{x} = M^{-1}\bar{x} + H^T R^{-1}z \]

Adding and subtracting \( H^T R^{-1}H\bar{x} \) to the RHS,

\[ (M^{-1} - H^T R^{-1}H)\hat{x} = (M^{-1} - H^T R^{-1}H)\bar{x} + H^T R^{-1}(z - H\hat{x}) \]
Kalman Filter

Now use

\[ P = M^{-1} - H^T R^{-1} H \]

to write the estimate for \( \hat{x} \) as

\[ \hat{x} = \bar{x} + P^{-1} H^T R^{-1} (z - H \bar{x}) \]

This is the least squares estimate for \( x \). It can be shown that \( P \) is the covariance of the error in the estimate, that is

\[ P = E[(\hat{x} - x)(\hat{x} - x)^T] \]
Now suppose that you would like to estimate the value of a variable as before. However now this variable $x_1$ is related to a previous variable $x_0$ by

$$x_1 = Ax_0 + B\mu$$

where

$$E[x_0] = \bar{x}_0$$

$$E[\mu] = \bar{\mu}$$

$$E[(\mu - \bar{\mu}_0)(\mu - \bar{\mu}_0)^T] = Q_0$$

and

$$E[(\hat{x}_0 - x_0)(\hat{x}_0 - x_0)^T] = P_0$$
This problem is easy since we have just solved it! The solution is given by

\[ \hat{x}_1 = \bar{x}_1 + P_1^{-1} H^T R^{-1} (z_1 - H \bar{x}_1) \]

where

\[ P_1 = M_1^{-1} - H^T R^{-1} H \]

The new wrinkle is that now

\[ \bar{x}_1 = A \bar{x}_0 + B \bar{\mu} \]

and

\[ M_1 = A P_0 A^T + B Q_0 B^T \]

This can be iteratively extended to handle the case where

\[ x_{k+1} = A x_k + B \mu \]