INDEPENDENT COMPONENTS ANALYSIS

From the basic setup we have

$$\mathbf{z} = G(WA\mathbf{u})$$

So that coversely,

$$\begin{aligned} \mathbf{u} &= A^{-1}W^{-1}G^{-1}(\mathbf{z}) && (1) \\ &= \Psi(\mathbf{z}) && (2) \end{aligned}$$

Now what we should have remembered from probability theory and linear algebra is that

$$p(\mathbf{z}) = \frac{p(\mathbf{u})}{|det(\mathbf{J}(\mathbf{u}))|}$$

where the RHS is evaluated at $\mathbf{u} = \Psi(\mathbf{z})$.

Now the entropy of $\mathbf{z}$ is given by

$$\begin{aligned} h(\mathbf{z}) &= -E[\log p(\mathbf{z})] && (3) \\ &= -E\left[\log\left(\frac{p(\mathbf{u})}{|det(\mathbf{J}(\mathbf{u}))|}\right)\right] && (4) \\ &= -KL_{p(\mathbf{u})||\ |detJ|} && (5) \end{aligned}$$

$$J_{ij} = \frac{\partial z_i}{\partial u_j}$$

but this can be expressed, using $\mathbf{z} = G(WA\mathbf{u})$ as

$$\begin{aligned} J_{ij} &= \sum_{k=1}^{m} \frac{\partial z_i}{\partial y_i} \frac{\partial y_i}{\partial x_k} \frac{\partial x_k}{\partial u_j} && (6) \\ &= \sum_{k=1}^{m} \frac{\partial z_i}{\partial y_i} w_{ik} a_{kj} && (7) \end{aligned}$$

So that finally,

$$J = DWA$$

where D is the diagonal matrix

$$D = diag\left(\frac{\partial z_1}{\partial y_1}, \frac{\partial z_2}{\partial y_2}, \ldots, \frac{\partial z_m}{\partial y_m}\right)$$

Now we are ready for the algorithm. Recall that

$$h(\mathbf{z}) = -E\left[\log\left(\frac{p(\mathbf{u})}{|det(\mathbf{J}(\mathbf{u})|}\right)\right]$$

So that

$$h(\mathbf{z}) = E\left[\log\left(\frac{|det(\mathbf{J}(\mathbf{u})|}{p(\mathbf{u})}\right)\right]$$

Thus *maximizing* entropy is equivalent to *maximizing* $\log(|det(\mathbf{J}(\mathbf{u})|)$.

So lets do this. Let $\Phi = \log(|det(\mathbf{J})$. Then since $J = DWA$,

$$\Phi = \log|det(A)| + log|det(W)| + \sum_{i=1}^{m} \log\frac{\partial z_i}{\partial y_i}$$

Now the only free parameters are W so we want wo find the W that maximize $\log(|det(\mathbf{J}(\mathbf{u})|)$.

So we'll change W by

$$\Delta W = \eta\frac{\partial\Phi}{\partial W}$$

Then

$$\begin{aligned} \frac{\partial\Phi}{\partial W} &= W^{-T} + \sum_{k=1}^{m}\frac{\partial}{\partial W}\log\frac{\partial z_i}{\partial y_i} && (8) \\ &= W^{-T} + (\mathbf{1} - 2\mathbf{z})\mathbf{x}^T && (9) \end{aligned}$$

The last step uses the fact that

$$z_i = \frac{1}{1+e^{-y_i}}$$

Almost done! The only issue remaining is that we don't want to evaluate W^{-T} each time.

The solution is to post multiply by W^TW. Then we have

$$\frac{\partial \Phi}{\partial W} = I + (\mathbf{1} - 2\mathbf{z})\mathbf{y}^T W$$