1 Pseudo-distributions

Recall that we can view the solution returned by $d$ levels of the Sum of Squares hierarchy as a function $\mu : \{0, 1\}^n \to \mathbb{R}$ such that there exists a formal "expectation" operator $\tilde{E}_\mu$ which acts on functions as follows:

$$\tilde{E}_\mu f = \sum_{x \in \{0, 1\}^n} \mu(x)f(x)$$

We say that $\mu$ is a degree $d$ pseudo-distribution if the above operator, called a pseudo-expectation, satisfies

$$\tilde{E}_\mu 1 = 1$$

and for all polynomials $p$ of degree at most $d/2$

$$\tilde{E}_\mu p^2 \geq 0$$

Show that pseudo-expectation satisfies a version of Cauchy-Schwarz. In particular if $\mu$ is a degree $d$ pseudo-distribution show that for polynomials $p, q$ of degree at most $d/2$,

$$\left(\tilde{E}_\mu pq\right)^2 \leq \left(\tilde{E}_\mu p^2\right)\left(\tilde{E}_\mu q^2\right)$$

2 3 XOR Lower Bound

Suppose we are given a system of $m$ equations over $n$ variables in $\{0, 1\}$ of the following form:

$$x_{i_1} + x_{i_2} + x_{i_3} = a_i \pmod{2}$$

where $a_i \in \{0, 1\}$. In this exercise we will show that for all $\epsilon > 0$ there exists such a system of equations such that every assignment $x \in \{0, 1\}^n$ satisfies at most $1/2 + \epsilon$ fraction of equations, but the optimal fraction of equations "satisfied" by $\Omega(n)$ levels of the Sum of Squares hierarchy is 1.

It will be convenient to view our system of equations as a 3-left-regular bipartite graph $G = (L \cap R, E)$ where there is a vertex in $L$ for each equation, a vertex in $R$ for each variable,
and we join each vertex \( v \) on the left to the three vertices on the right which correspond to the variables in the equation corresponding to \( v \). Finally, we can write down the \( a_i \)'s as a vector \( a \in \{0, 1\}^m \).

First we show that a randomly selected system is with high probability not much more than \( 1/2 \) satisfiable.

### 2.1 Presentation 1 - Soundness

Fix \( \epsilon > 0 \), and let \( n \) be the number of variables. Suppose our system has \( m > 9n/\epsilon^2 \) equations. Show that if we select an \( a \in \{0, 1\}^m \) uniformly at random, with probability \( 1 - o_n(1) \) no assignment \( x \in \{0, 1\}^n \) satisfies more that \( 1/2 + \epsilon \) fraction of equations.

Let \( G \) be a bipartite graph which is left \( d \)-regular. For a set of left vertices \( S \) we denote its set of neighbors \( \Gamma(S) \). Suppose that for any subset \( S \) of left vertices of size at most \( s \), that \( \Gamma(S) \geq \alpha |S| \). We call such a graph a \((d, s, \alpha)\) expander.

### 2.2 Presentation 2 - Expansion of Random Instances

Consider the following probabilistic construction of a system of equations. Independently for each triple \( x_i, x_j, x_k \), we include them together in an equation with probability \( p/n^2 \) where \( p \) is some number which depends on \( \epsilon \) but not \( n \). Show that for some choice of constant \( \gamma \in [0, 1] \), that a graph sampled in this manner is a \((3, \gamma n, 1.7)\) expander with probability at least 0.9.

The upshot of the previous two exercises is that a randomly chosen set of equations is not very satisfiable, and its induced graph is an expander. Now, we will construct a pseudodistribution \( \mu \) of degree \( d = \gamma n/10 \) which "satisfies" all equations.

Given an equation \( x_{i1} + x_{i2} + x_{i3} = a_i \) we can encode it as a polynomial \( (1 - 2a_i)(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3}) \), which evaluates to 1 precisely when the equation is satisfied. Let \( \chi_S = \prod_{i \in S}(1 - 2x_i) \) for \( S \subset [n] \), and let \( \chi_{\emptyset} \) be identically 1. Now, the space of polynomials (with boolean inputs) of degree \( \leq d \) is spanned by polynomials \( \{\chi_S : |S| \leq d\} \), so to construct a degree \( d \) pseudodistribution \( \mu \) we need only specify the value of \( \mathbb{E}_\mu \) on these polynomials.

### 2.3 Presentation 3 - Pseudo-Distribution Construction and Local Consistency

We will construct the pseudodistribution \( \mu \) in the following way. First, we set

\[
\mathbb{E}_\mu \chi_{\emptyset} = 1
\]
Next, since it must satisfy \((1 - 2a_i)(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3})\) for each equation \(x_{i1} + x_{i2} + x_{i3} = a_i\), we set

\[
\tilde{E}_\mu(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3}) = (1 - 2a_i)
\]

Now, we continue in the following manner as long as possible: Pick a pair of subsets with \(|S|, |T| \leq d\) such that \(\tilde{E}_\mu \chi_S\) and \(\tilde{E}_\mu \chi_T\) have been set. If it hasn’t been set yet and \(|S \triangle T| \leq d\), assign

\[
\tilde{E}_\mu \chi_{S \triangle T} = (\tilde{E}_\mu \chi_S)(\tilde{E}_\mu \chi_T)
\]

Else if the left hand side has been set to a value other than the right hand side this process ends and fails. Finally if we can no longer continue this process, for all unassigned subsets \(S\) of size at most \(d\) we set

\[
\tilde{E}_\mu \chi_S = 0
\]

Show that if the graph associated with our system of equations is a \((3, 10n, 1.7)\) expander, that the above process never fails.

### 2.4 Presentation 4 - Pseudo-Distribution is Positive Semidefinite

The last step is to show that we’ve constructed \(\mu\) as a valid pseudo-distribution. In particular prove that for a polynomial \(p\) with degree at most \(d/2\),

\[
\tilde{E}_\mu p^2 \geq 0
\]

Conclude that after \(\Omega(n)\) rounds of Sum of Squares heirarchy the SDP has an integrality gap of 2. In particular we’ve shown that Sum of Squares cannot outperform the trivial algorithm of a random assignment in polynomial time.