Workshop I - Label Cover, Unique Label Cover, Linear Programming

The workshop will be held on Monday, September 10th. Each student must submit a written solution to 1 problem of their choice at the beginning of class on the day of the workshop. Specify which problem you chose at the top of your submission. Alternatively, a student may volunteer to present a problem to the class in the workshop. The workshop will have 5 presenters, each presenting one problem in the order given here. Each presentation should be about 5 minutes. Register to present on September 5, upon an announcement from the TA.

1 Label Cover

An instance of Label Cover consists of a bipartite graph \((L, R, E)\), a finite set of labels \(\Sigma_L\) for the left vertices and a finite set of labels \(\Sigma_R\) for the right vertices, as well as a family of functions \(\{f_e : \Sigma_L \rightarrow \Sigma_R \cup \{\bot\} | e \in E\}\). A labelling is a pair of functions \(l_L : L \rightarrow \Sigma_L\) and \(l_R : R \rightarrow \Sigma_R\). An edge \(e = (u, v)\) is satisfied if \(f_e(l_L(u)) = l_R(v)\).

Our objective is to find \(l_L, l_R\) so as to maximize the fraction of edges satisfied. (Each below counts as 1 problem)

1.1 \(\frac{1}{|\Sigma_R|}\)-Approximation

In your presentation you should define the Label Cover problem and discuss the connection to Max-k-CSP. Last, present an efficient algorithm which satisfies \(\frac{1}{|\Sigma_R|}\) fraction of the optimal number of constraints.

1.2 \(\frac{d}{n}\)-Approximation

Assume we are given an instance of label cover where there are \(n\) right vertices, and each of the left vertices has degree \(d\). Present an efficient algorithm which satisfies \(\frac{d}{n}\) fraction of the optimal number of constraints. Discuss the special case where the graph is a complete bipartite graph (i.e., \(d = n\)).

1.3 \(\frac{1}{d}\)-Approximation \(\rightarrow (|\Sigma_R|n)^{-1/3}\)-Approximation

Assume our instance is as in part 2. Present an algorithm which satisfies \(\frac{1}{d}\) fraction of the optimal number of constraints. Combine the three previous algorithms, and present an algorithm which satisfies \((|\Sigma_R|n)^{-1/3}\) fraction of the optimal number of constraints. Hint: Use the AM-GM inequality, which states that Arithmetic Mean ¿ Geometric Mean

2 Unique Label Cover

An instance of Unique Label Cover consists of a bipartite graph \((L, R, E)\), a finite set of labels \(\Sigma\), and a set of 1-1 functions \(\{f_e : \Sigma \rightarrow \Sigma | e \in E\}\). A labelling is a function \(l : L \cup R \rightarrow \Sigma\). An edge \(e = (u, v)\) is satisfied if \(f_e(l(u)) = l(v)\). Our objective is to find a labelling so as to maximize the number of edges satisfied.

In your presentation present Unique Label Cover and show how to efficiently decide if a given instance of Unique Label Cover has a labelling which satisfies all the constraints.

3 Linear Programming for Label Cover

Recall the definition of the Label Cover problem as in Problem 1. Formulate the Label Cover problem as an integer linear program. That is, explain how to convert an instance of label cover into an optimization problem with linear objective function and linear constraints, with all variables constrained to be in \(\{0, 1\}\). Be sure to justify why your objective function and constraints accurately model the original label cover problem. Discuss the integrality gap between your integer program and its LP relaxation.