Workshop IV - ∑um of Squares Workshop

September 25, 2018

The workshop will be held on October 1st. There are five problems and the approximate time you should spend presenting each problem is designated.

1 Pseudo-distributions (5 minutes)

Recall that we can view the solution returned by $d$ levels of the Sum of Squares hierarchy as a function $\mu : \{0, 1\}^n \rightarrow \mathbb{R}$ such that there exists a formal “expectation” operator $\mathbb{E}_\mu$ which acts on functions as follows:

$$\mathbb{E}_\mu f = \sum_{x \in \{0, 1\}^n} \mu(x) f(x)$$

We say that $\mu$ is a degree $d$ pseudo-distribution if the above operator, called a pseudo-expectation, satisfies

$$\mathbb{E}_\mu 1 = 1$$

and for all polynomials $p$ of degree at most $d/2$

$$\mathbb{E}_\mu p^2 \geq 0$$

Show that pseudo-expectation satisfies a version of Cauchy-Schwarz. In particular if $\mu$ is a degree $d$ pseudo-distribution show that for polynomials $p, q$ of degree at most $d/2$,

$$(\mathbb{E}_\mu pq)^2 \leq (\mathbb{E}_\mu p^2)(\mathbb{E}_\mu q^2)$$

2 3 XOR Lower Bound

Suppose we are given a system of $m$ equations over $n$ variables in $\{0, 1\}$ of the following form:

$$x_{i1} + x_{i2} + x_{i3} = a_i \pmod{2}$$

where $a_i \in \{0, 1\}$. In this exercise we will show that for all $\epsilon > 0$ there exists such a system of equations such that every assignment $x \in \{0, 1\}^n$ satisfies at most $1/2 + \epsilon$ fraction of equations, but the optimal fraction of equations “satisfied” by $\Omega(n)$ levels of the Sum of
Squares hierarchy is 1.

It will be convenient to view our system of equations as a 3-left-regular bipartite graph $G = (L \cup R, E)$ where there is a vertex in $L$ for each equation, a vertex in $R$ for each variable, and we join each vertex $v$ on the left to the three vertices on the right which correspond to the variables in the equation corresponding to $v$. Finally, we can write down the $a_i$s as a vector $a \in \{0, 1\}^m$.

First we show that a randomly selected system is with high probability not much more than $1/2$ satisfiable.

2.1 Presentation 1 - Soundness (10 minutes)

Fix $\epsilon > 0$, and let $n$ be the number of variables. Suppose our system has $m > 9n/\epsilon^2$ equations. Show that if we select an $a \in \{0, 1\}^m$ uniformly at random, with probability $1 - o_n(1)$ no assignment $x \in \{0, 1\}^n$ satisfies more than $1/2 + \epsilon$ fraction of equations.

Let $G$ be a bipartite graph which is left $d$-regular. For a set of left vertices $S$ we denote its set of neighbors $\Gamma(S)$. Suppose that for any subset $S$ of left vertices of size at most $s$, that $\Gamma(S) \geq \alpha |S|$. We call such a graph a $(d, s, \alpha)$ expander.

2.2 Presentation 2 - Expansion of Random Instances (10 minutes)

Consider the following probabilistic construction of a system of equations. Independently for each triple $x_{i1}, x_{i2}, x_{i3}$, we include them together in an equation with probability $p/n^2$ where $p$ is some number which depends on $\epsilon$ but not $n$. Show that for some choice of constant $\gamma \in [0, 1]$, that a graph sampled in this manner is a $(3, \gamma n, 1.7)$ expander with probability at least 0.9.

The upshot of the previous two exercises is that a randomly chosen set of equations is not very satisfiable, and its induced graph is an expander. Now, we will construct a pseudo-distribution $\mu$ of degree $d = \gamma n/10$ which “satisfies” all equations.

Given an equation $x_{i1} + x_{i2} + x_{i3} = a_i$ we can encode it as a polynomial $(1 - 2a_i)(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3})$, which evaluates to 1 precisely when the equation is satisfied. Let $\chi_S = \prod_{i \in S}(1 - 2x_i)$ for $S \subset [n]$, and let $\chi_{\emptyset}$ be identically 1. Now, the space of polynomials (with boolean inputs) of degree $\leq d$ is spanned by polynomials $\{\chi_S : |S| \leq d\}$, so to construct a degree $d$ pseudo-distribution $\mu$ we need only specify the value of $\bar{E}_\mu$ on these polynomials.
2.3 Presentation 3 - Pseudo-Distribution Construction and Local Consistency (15 minutes)

We will construct the pseudo-distribution $\mu$ in the following way. First, we set

$$\tilde{E}_\mu \chi_{\emptyset} = 1$$

Next, since it must satisfy $(1 - 2a_i)(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3})$ for each equation $x_{i1} + x_{i2} + x_{i3} = a_i$, we set

$$\tilde{E}_\mu (1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3}) = (1 - 2a_i)$$

Now, we continue in the following manner as long as possible: Pick a pair of subsets with $|S|, |T| \leq d$ such that $\tilde{E}_\mu \chi_S$ and $\tilde{E}_\mu \chi_T$ have been set. If it hasn’t been set yet and $|S \triangle T| \leq d$, assign

$$\tilde{E}_\mu \chi_{S \triangle T} = (\tilde{E}_\mu \chi_S)(\tilde{E}_\mu \chi_T)$$

Else if the left hand side has been set to a value other than the right hand side this process ends and fails. Finally if we can no longer continue this process, for all unassigned subsets $S$ of size at most $d$ we set

$$\tilde{E}_\mu \chi_S = 0$$

Show that if the graph associated with our system of equations is a $(3, 10n, 1.7)$ expander, that the above process never fails.

2.4 Presentation 4 - Pseudo-Distribution is Positive Semidefinite (15 minutes)

The last step is to show that we’ve constructed $\mu$ as a valid pseudo-distribution. In particular prove that for a polynomial $p$ with degree at most $d/2$,

$$\tilde{E}_\mu p^2 \geq 0$$

Conclude that after $\Omega(n)$ rounds of Sum of Squares heirarchy the SDP has an integrality gap of 2. In particular we’ve shown that Sum of Squares cannot outperform the trivial algorithm of a random assignment in polynomial time.