CS388C: Combinatorics and Graph Theory — Problem Set No. 6
David Zuckerman — due: Nov. 13, 2007

1. The designs from projective planes can be generalized to higher dimensions. A d-dimensional projective geometry over a field $F = \mathbb{F}_q$ has as points the one-dimensional subspaces in F^{d+1}. The points of the design are the points of the geometry, and the blocks of the design are the ℓ-dimensional subspaces of the geometry (which are the $\ell+1$-dimensional subspaces of F^{d+1}), where $0 < \ell < d$. Consider the case $d = 3$ and $\ell = 2$. Show that this is a design, and compute its parameters (v, k, λ), as well as the number of blocks.

2. Let p be prime and $L_1, \ldots, L_m$ be sets of integers, $|L_i| = s$. Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$ is a family of subsets of $[n]$ such that

   (a) $|A_i| \notin L_i \pmod p$, $i \in [m]$;

   (b) $|A_i \cap A_j| \in L_i \pmod p$, $1 \leq j < i \leq m$.

   Show that
   $$m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}.$$

The solutions of all problems below involve codes. For any prime power q and any $\epsilon > 0$, you may assume that there are efficiently constructible generator matrices of $[n, k, d = (1 - 1/q - \epsilon)n]_q$ codes, where $n = \text{poly}(k, 1/\epsilon)$.

3. Construct k binary random variables $X_1, \ldots, X_k$ on a probability space of size $\text{poly}(k, 1/\epsilon)$ such that for any non-empty $S \subseteq [k]$,
   $$|\Pr[\oplus_{i \in S} X_i = 1] - 1/2| \leq \epsilon.$$

   Hint: you may assume that the suitable code contains the all 1's vector.

4. The goal of this problem is, for parameters $\ell \leq q$, to construct, for every m, $O(\log m)$ hash functions $h_i : [m] \to [q]$, such that for any subset $S \subseteq [m]$ of size ℓ, at least one of the h_i is injective (1-1) on S. For example, if $q = \ell = 2$, we can take $h_i(x)$ to be the ith bit of x. For any constant ℓ, give a constant q and a construction of such hash functions for every m.

5. A group of n players first agree on a strategy and then play the following game. A hat is placed on each player's head that is red or blue, each with probability half and independent of the other hats. Each player can see the hats of all other players, but not his own. Next, each player is given the option to guess the color of his hat; he may either make no guess or guess "red" or "blue." When making his guess, a player may

not see other players' guesses. The players win if at least one player guesses and all guesses are correct.

Show a strategy for the players which achieves success probability $1 - 1/(n+1)$ if $n = 2^t - 1$ for some integer $t \geq 2$.

Hint: Think about directing the edges of the n-dimensional hypercube. It's helpful to first show how to achieve probability $3/4$ with $n = 3$.