

1. Let  $G$  be the bipartite line-point incidence graph of a projective plane over  $\mathbb{F}_q$ , i.e., the lines are on the left, points on the right, and a line is adjacent to a point iff the point lies on the line. Let  $A$  be the adjacency matrix of  $G$ . Because  $G$  is a bipartite  $d$ -regular graph (what is  $d$ ?),  $A$  has eigenvalues  $\pm d$ . Show that all other eigenvalues have absolute value at most  $\sqrt{d}$ . (Hint: consider  $A^2$ .)
2. Let  $G = (V, E)$  be an expander graph (directed or undirected) on  $n = |V|$  nodes, such that every set of size  $n/2$  has at least  $n/2 + \alpha n$  neighbors. For any  $k$ , let  $S_1, \dots, S_k$  be arbitrary subsets of  $V$  such that for all  $i$ ,  $|S_i| \geq (1 - \alpha)n$ . Show that there exists a path  $v_1, \dots, v_k$  in  $G$  such that for all  $i$ ,  $v_i \in S_i$ .
3. Let  $G$  be a  $d$ -regular graph on  $n$  vertices with adjacency matrix  $A$ . Let the smallest eigenvalue of  $A$  be  $\lambda_n$  (so  $\lambda_n < 0$ ). Let  $e(S)$  denote the number of edges with both endpoints in  $S$ .

(a) Show that for any set  $S$  of size  $s$ ,

$$e(S) \geq \frac{1}{2} \left( \frac{ds^2}{n} + \lambda_n s \left( 1 - \frac{s}{n} \right) \right).$$

(b) Show that the size of the largest independent set is at most  $-n\lambda_n/(d - \lambda_n)$ .

4. (20 points) Consider a bipartite graph with  $n$  left vertices,  $m$  right vertices, and left degree  $d$ , such that any subset  $S$  on the left of size at most  $k$  has at least  $(1 - \epsilon)d|S|$  neighbors.

(a) Prove that for a set  $S$  of size  $s \leq k/2$ , at most  $s/2$  vertices outside  $S$  have at least  $\delta d$  neighbors in  $\Gamma(S)$ , for  $\delta = O(\epsilon)$ .

We can use such expanders for data structures. Suppose we want to store a small subset  $S$  of a large universe  $[n]$  such that we can test membership in  $S$  by probing just 1 bit of our data structure. Storing the characteristic vector of  $S$  requires  $n$  bits of storage. There are hash-based data structures which require storing only  $O(|S|)$  words, each of  $O(\log n)$  bits, but testing membership requires reading an entire word (rather than just one bit). Our data structure will consist of  $m$  bits, which we think of as a  $\{0, 1\}$ -assignment to the right vertices of our expander. This assignment will satisfy:

Property A: For all left vertices  $x$ , all but a  $\delta = O(\epsilon)$  fraction of the neighbors of  $x$  are assigned the value  $\chi_S(x)$ , where  $\chi_S(x) = 1$  iff  $x \in S$ .

(b) Show that for an assignment satisfying Property A, we can probabilistically test membership in  $S$  with error probability  $\delta$  by reading just one bit of the data structure.

(c) Show that an assignment satisfying Property A exists provided  $|S| \leq k/2$ . (Hint: first assign 1 to all neighbors of  $S$  and 0 to all non-neighbors; then correct the errors.)