Definition 1. Let $A = (Q, \Sigma, \delta, q_I, Acc)$ be an $\omega$-automaton. A run of $A$ on an $\omega$-word $\alpha = a_1a_2\ldots \in \Sigma^\omega$ is an infinite state sequence $\varrho = \varrho(0)\varrho(1)\ldots \in Q^\omega$ such that the following conditions hold:

1. $\varrho(0) = q_I$
2. $\varrho(i) \in \delta(\varrho(i-1), a_i)$ for $i \geq 1$ if $A$ is non-deterministic,
   $\varrho(i) = \delta(\varrho(i-1), a_i)$ for $i \geq 1$ if $A$ is deterministic.

Büchi Acceptance: An $\omega$-automaton with acceptance component $F \subseteq Q$. A word $\alpha \in \Sigma^\omega$ is accepted by $A$ iff there exists a run $\varrho$ of $A$ on $\alpha$ satisfying the condition:

$$\text{Inf}(\varrho) \cap F \neq \emptyset$$

The language accepted by the automaton $A$ is $L(A) = \{ \alpha \in \Sigma^\omega | A \text{ accepts } \alpha \}$.

1. In $\omega$-automaton with the büchi acceptance prove that non-determinism is strictly more expressive than determinism. (That is there exists a language $L_1$ which can be recognized by a non-deterministic büchi automaton, but not by any deterministic büchi automaton.)

$\omega$-regular: A language $L$ is said to be $\omega$-regular if it is accepted by some büchi automaton.

Definition 2 (Limit Languages). Let $L$ be a finite-regular language. Define $\hat{L}$ to be the $\omega$-language consisting of all the words that have infinitely many prefixes in $L$.

2. Let $A$ be a deterministic büchi automaton and let $L_f(A)$ be the language of finite words accepted by $A$ when treated as a finite automaton and let $L(A)$ be the language accepted by $A$ as a büchi automaton. Prove that

$$L(A) = \hat{L_f(A)}.$$

3. Show that the emptiness problem is decidable for büchi automata.
The following set of problems will help you prove that \(\omega\)-regular languages are closed under union, intersection, complementation.

4. **Show how to construct a Büchi automaton accepting** \(L_1 \cap L_2\) **given automaton accepting** \(L_1\) **and** \(L_2\).

5. **Show how to construct a deterministic Büchi automaton accepting** \(L_1 \cup L_2\) **and** \(L_1 \cap L_2\) **from deterministic automaton accepting** \(L_1\) **and** \(L_2\).

Intersection and union are relatively easy. Next we show that \(\omega\)-regular languages are closed under complementation.

For the complementation of regular languages Myhill-Nerode congruences was one of the tools available to us. We establish a similar notion of congruences to capture accepting runs over infinite words.

**Definition 3.** With each Büchi automaton \(A\), we associate a relation \(\equiv_A\) over \(\Sigma^*\) given by

\[
x \equiv_A y \triangleq \forall q, q'. q \xrightarrow{x} q' \iff q \xrightarrow{y} q' \land \forall q, q'. q \xrightarrow{y} q' \iff q \xrightarrow{y} q'
\]

where \(q \xrightarrow{z} q'\) means that there is run from \(q\) to \(q'\) on the word \(x\) that passes through some state in \(F\). \(^1\)

6. **Show that** \(\equiv_A\) **is a congruence.**

7. **Let** \(U\) **and** \(V\) **be regular languages. Then show that** \(U.V^\omega\) **is** \(\omega\)-regular.

From (6) and (7) if \(\equiv_A\) has \(N\) equivalence classes, then each of the \(\omega\)-language obtained as \([u].[v]^\omega\) is either completely contained in \(L(A)\) or in \(\overline{L(A)}\). Unlike the case of finite words where it is trivial that each word of \(\Sigma^*\) lies in some equivalence class of \(\equiv_A\), it is not clear that every \(\omega\)-word is an element of \([u].[v]^\omega\) for some \(u\) and \(v\). This needs proof. First prove a general result about monoids.

8. **Let** \(F\) **be the free monoid\(^2\) over** \(\Sigma\) **(possibly infinite).** \(M\) **be a finite monoid and let** \(h\) **be a homomorphism from** \(F\) **to** \(M\). **Let** \(\alpha = a_1a_2\ldots\) **be any** \(\omega\)-word over \(\Sigma\). **Then, there are elements** \(s\) **and** \(e\) **in** \(M\) **such that** \(e.e = e\) **and** \(\alpha \in h^{-1}(s). (h^{-1}(e))^\omega\).

(Hint: Use the infinite version of the Ramsey’s theorem. However it is also possible to give a direct argument.)

9. **Use** (8) **and** \(\Sigma_A\) **to find the complement of an** \(\omega\)-language \(L\). **(Hint: Use the quotient monoid.)**

10. **Prove that a language** \(L\) **over** \(\Sigma^\omega\) **is** \(\omega\)-regular iff there is a homomorphism** \(h\) **from** \(\Sigma^*\) **to a finite monoid** \(M\) **and a collection** \(X\) **of linked pairs (Pairs** \((s, e)\) **such that** \(s.e = s\) **and** \(e.e = e\) **over** \(M\) **such that** \(L = \bigcup_{(s, e) \in X} h^{-1}(s). (h^{-1}(e))^\omega\).**

\(^1\)I shall write \([x]\) for \([x]_{\equiv_A}\).

\(^2\)For a definition of free monoid on \(\Sigma\) refer to Automata and Computability by Dexter Kozen.
References

Dominique Perrin and Jean-Eric Pin: Infinite words: Automata, Semi-groups, Logic and Games LNCS 2500

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