Mathematical Induction

How to prove:
For every integer \( n \geq 1 \), \( P(n) \) holds.

Mathematical induction is a method for proving a statement holds for all positive integers (or for all integers \( n \geq n_0 \), for some \( n_0 \in \mathbb{Z} \)).

Example: Prove that \( 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \) for all \( n \geq 1 \).

In this example, \( P(n) \) is: \( 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \).
We want to show:
For all integers \( n \geq 1 \), \( P(n) \).

\( P(1) \) says: \( 1 = 1^2 \).
\( P(2) \) says: \( 1 + 3 = 2^2 \).
\( P(3) \) says: \( 1 + 3 + 5 = 3^2 \).
...
How does Mathematical Induction Work?

To prove that, for some P, P(n) is true for all integers \( n \geq 1 \):

The Principle of Mathematical Induction says that you need to:
1. Prove that P(1) is true. (base case)
2. Prove that for every \( k \geq 1 \), if P(k) is true, then P(k+1) is also true.

How to do step 2:

**AKA:** direct proof of a universally quantified implication
a. Assume that P(k) is true for some arbitrarily chosen \( k \geq 1 \) (induction hypothesis)
b. Show from this assumption that P(k+1) is also true (induction step).

First you prove the base case, that is, that P(1) is true.
If you prove the induction step, then P(1) is true implies P(2) is true, and P(2) being true implies P(3) is true, and P(3) being true implies P(4) being true, etc.
First Induction Example

**Example:** Prove that \(1 + 3 + 5 + \ldots + (2n - 1) = n^2\) for all integers \(n \geq 1\).

**Proof:** (in class)

**Note:** In the induction step, re-write \(P(k+1)\) in terms of \(P(k)\) so that you can use the assumption that \(P(k)\) is true.

**Reminder:** In your proofs, never, ever write down that two quantities are equal **before** you have proved they are equal!
Induction Examples

**Exercise:** Prove that \(1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}\) for all integers \(n \geq 1\).

**Note:** There is nothing magic about starting at \(n = 1\). We can start at any integer \(n_0\) and prove that: For any integer \(n \geq n_0\), \(P(n)\). Our base case is \(P(n_0)\). We assume that \(P(k)\) is true for some \(k \geq n_0\), and show that \(P(k+1)\) is true.

**Theorem:** For every integer \(n \geq 0\), \(2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1\).

**Theorem:** For every integer \(n > 0\), \(3|(n^3 - n)\).
Strong Mathematical Induction

Recall The Principle of Mathematical Induction:
If \( P(n_0) \) is true, and for each integer \( k \geq n_0 \), \( P(k+1) \) is true whenever \( P(k) \) is true, then \( P(n) \) is true for all \( n \geq n_0 \).

**Stronger Form:** To show \( P(n) \) is true for all \( n \geq n_0 \).
1. Show \( P(n_0) \) is true (base case).
2. Assume \( P(n_0), P(n_0 + 1), \ldots, P(k) \) are true for some \( k \geq n_0 \) (induction hypothesis).
3. Show \( P(k+1) \) is true.

**Example:** For any integer \( n > 1 \), \( n \) is divisible by a prime.

**Example:** The Fibonacci numbers \( f_0, f_1, f_2, \ldots \) are defined by \( f_0 = f_1 = 1 \) and \( f_n = f_{n-1} + f_{n-2} \) for all \( n \geq 2 \). Prove that for all \( n \geq 0 \), \( f_n \leq \left(\frac{7}{4}\right)^n \).