Recurrence Relations

Plan for Today:

- Applications of recurrence relations
- Solving recurrence relations
- Divide-and-conquer algorithms
Introduction to Recurrence Relations

A recurrence relation is a sequence which is defined in terms of itself.

Note: Many algorithms, divide and conquer algorithms in particular, have a time complexity function that is easily modeled by a recurrence relation.

Examples:

1. \( a_n = a_{n-1} + 1, \ a_1 = 1 \)
   Note that \( a_n = a_{n-1} + 1 = a_{n-2} + 2 = \ldots = a_1 + (n - 1) = n \).

2. \( a_n = 2 \times a_{n-1}, \ a_1 = 1 \)
   Note that \( a_n = 2a_{n-1} = 2^2a_{n-2} = \ldots = 2^i a_{n-i} = 2^{n-1} a_1 = 2^{n-1} \)

Note: We also find recurrences useful for solving counting problems.
Modeling with Recurrence Relations: Tower of Hanoi

**Input:** Tower of $n$ disks, stacked in order of increasing size (top to bottom) on one of 3 pegs

**Goal:** Transfer entire tower to one of the other 3 pegs, one disk at a time, never moving a larger disk onto a smaller disk.

Call the 3 pegs, in order from left to right, Peg 1, Peg 2, and Peg 3. The disks are originally on Peg 1, and we will move them to Peg 3.

**Algorithm:**

1. Move top $n - 1$ disks from Peg 1 to Peg 2 (using Peg 3 as the intermediate peg).
2. Move the bottom disk from Peg 1 to Peg 3.
3. Move $n - 1$ disks from Peg 2 to Peg 3 using Peg 1 as intermediate peg.

**Psuedocode:**

```plaintext
Hanoi(n, Peg 1, Peg 2, Peg 3)
{
    if n == 0 return
    Hanoi(n-1, Peg 1, Peg 3, Peg 2)
    move 1 disk from Peg 1 to Peg 3
    Hanoi(n-1, Peg 2, Peg 1, Peg 3)
}
```
Tower of Hanoi

This Hanoi function is recursive, since it calls itself repeatedly for decreasing values of $n$ (until the condition $n == 0$ is satisfied and the function terminates).

Let $T_n =$ minimum number of moves needed to solve the puzzle for $n$ disks.
Then $T_0 = 0$ and $T_1 = 1$.
Define $T_n$ by:
$T_0 = 0$
$T_n = 2T_{n-1} + 1$

**Solution:** in class
**Time Complexity Functions and Recurrence Relations**

Recurrence relations can give us an easy way to determine big-O complexity of a recursive function. Consider the following generic algorithm:

```c
SomeRecFn(float A[], int left, int right) {
    // Continue as long as left < right
    int mid = (left + right) / 2;
    if (left < right) {
        SomeRecFn(A, left, mid);
        SomeRecFn(A, mid+1, right);
        OtherJunk(A, left, mid, right); // n operations
    }
}
```

**Recurrence Relation:** Let $T_n$ be the time to execute the function on an array of size $n$ (i.e. left-right = $n$). Note that $T(1) = 1$. So we have:

- $T_n = 2T_{n/2} + n$
- $T_1 = 1$

**Solution:** in class
Counting and Recurrence Relations

Example: Determine a recurrence relation that describes the number of bit strings of length \( n \) that do not have two consecutive 0s.

Solution: Let \( a_n \) be the number of bit strings of length \( n \) that do not have two consecutive 0s. Assume \( n \geq 3 \).

The number of length \( n \) bit strings that start with 0 and do not contain two consecutive 0s: \( a_{n-2} \) (since the second bit must be 1)

The number of length \( n \) bit strings that start with 1 and do not contain two consecutive 0s: \( a_{n-1} \)

By the sum rule, \( a_n = a_{n-1} + a_{n-2} \). The initial conditions are \( a_1 = 2 \) and \( a_2 = 3 \).

So, \( a_5 = a_4 + a_3 = a_3 + a_2 + a_2 + a_1 = a_2 + a_1 + a_2 + a_2 + a_1 = 3 + 2 + 3 + 3 + 2 = 13 \).

Exercise: Find a recurrence relation for the number of bit strings of length \( n \) that contain three consecutive 0s. What are the initial conditions? How many bit strings of length 7 contain three consecutive 0s?
Another Recurrence Relation Example

A zookeeper has $n$ cages lined up and two indistinguishable lions. The lions must be put in separate cages, and the cages cannot be adjacent. Write a recurrence relation to describe $L(n)$, the number of ways that the two lions can be placed in cages. Remember to include the initial conditions.

Solution: in class
First Order Linear Recurrence Relations

Definition: A geometric progression is an infinite sequence of numbers, such as 5, 25, 125, ... where the division of each term other than the first by the previous term is a constant called the common ratio. (For the given sequence, the common ratio is 5).

So if $a_0, a_1, a_2, ...$ is a geometric progression, then $a_1/a_0 = a_2/a_1 = ... = a_n/a_{n-1} = r$, where $r$ is the common ratio.

Note: Knowing that $a_n = 5a_{n-1}$ (ie., knowing the common ratio) does not uniquely define a geometric progression, unless you also know a term in the sequence. The following sequences all have a common ratio of 5: 5, 25, 125, ...
2, 10, 50, ...
3, 15, 45, ...

Example: $a_n = 5a_{n-1}$ for $n \geq 1$

$a_0 = 5$

This sequence specifies 5, 25, 125, ....

Example: The recurrence relation $a_n = 5a_{n-1}$ for $n \geq 1$

$a_0 = 2$

specifies the sequence 2, 10, 50, 250, ....
First Order Linear Recurrence Relations

**Definition:** A recurrence relation is a **first order** recurrence relation if $a_n$ depends only on the previous term in the sequence.

A **first order linear homogeneous recurrence relation** is a first order recurrence relation which has constant coefficients and each subscripted term is raised to the first power. **General form:** $a_n = da_{n-1}$ where $d$ is a constant.

**Terminology:** Values like $a_0$ which are given with a recurrence relation are often called **boundary conditions**.

**Example:** Solve the recurrence relation

- $a_n = 4a_{n-1}$ for $n \geq 1$
- $a_0 = 3$

**Theorem:** The unique solution of the recurrence relation

- $a_n = da_{n-1}$, for $n \geq 1$
- $a_0 = A$

where $d$ is a constant, is given by

- $a_n = Ad^n$, $n \geq 0$.

**Example:** Solve the recurrence relation:

- $a_n = 7a_{n-1}$ for $n \geq 1$
- $a_2 = 98$

**Solution:** The trick here is that we are not given $a_0$. So the solu-
tion as the form \( a_n = a_0 7^n \). Since \( a_2 = 98 = a_0 \times 7^2 \), it follows that \( a_0 = 2 \). So \( a_n = 2 \times 7^n \).

**Note:** The recurrence relation \( a_n = da_{n-1} \) is **linear** because each subscripted term is raised to the first power. We don’t have \( a_n a_{n-1} \) or \( a^2_n \) appearing in a linear recurrence relation.

**Example:** What is \( a_6 \) if
\[
a^2_n = 5a^2_{n-1}, \text{ where } a_n > 0 \text{ for } n \geq 0 \text{ and } a_0 = 2.
\]

**Solution:** If we substitute \( b_n = a^2_n \), we get the new relation \( b_n = 5b_{n-1} \) for \( n \geq 0 \) and \( b_0 = 4 \), which is a linear relation with solution \( b_n = 4 \times 5^n \). So \( a_n = 2(\sqrt{5})^n \) for \( n \geq 0 \), and \( a_6 = 2(\sqrt{5})^6 = 250 \).
First Order Linear Recurrence Relations

**Definition:** The general first order linear recurrence relation with constant coefficients has the form
\[ a_n + ca_{n-1} = f(n), \ n \geq 0, \] where \( c \) is a constant and \( f(n) \) is a function on set \( \mathbb{N} \). When \( f(n) = 0 \) for all \( n \in \mathbb{N} \), the relation is **homogeneous**; otherwise, it is called **non-homogeneous**.

**Example:** A non-homogeneous linear first-order recurrence relation:
\[ a_n = a_{n-1} + (n - 1) \text{ for } n \geq 2 \]
\[ a_1 = 0 \]
(In this example, \( f(n) = n - 1 \)).
The Second Order Linear Homogeneous Relation with Constant Coefficients

**Definition:** Let \( k \in \mathbb{Z}^+ \) and \( C_0, C_1, ..., C_k \) be non-zero real numbers. Then
\[
C_0a_n + C_1a_{n-1} + c_2a_{n-2} + ... + C_k a_{n-k} = f(n), \quad n \geq k,
\]
is a **linear recurrence relation with constant coefficients of order** \( k \). When \( f(n) = 0 \) for all \( n \geq 0 \), the relation is **homogeneous**; otherwise, it is **non-homogeneous**.

We will now focus on the homogeneous relation of order 2:
\[
C_0a_n + C_1a_{n-1} + C_2a_{n-2} = 0, \quad n \geq 2.
\]

We will look for a solution of the form \( a_n = cr^n \) where \( c \neq 0 \) and \( r \neq 0 \).

Substituting \( a_n = cr^n \) into our relation above, we get:
\[
C_0cr^n + C_1cr^{n-1} + C_2cr^{n-2} = 0.
\]

Since \( c, r \neq 0 \), we get the quadratic equation known as the **characteristic equation**:
\[
C_0r^2 + C_1r + C_2 = 0.
\]

The roots \( r_1, r_2 \) of this equation are called the **characteristic roots**, and there are 3 cases:

1. \( r_1, r_2 \) are non-equal real numbers
2. \( r_1, r_2 \) form a complex conjugate pair
3. \( r_1, r_2 \) are real numbers and \( r_1 = r_2 \)

**Second Order Linear Homogeneous Recurrence Relations**

**Case 1:** Distinct Real Roots

**Example:**
\[ a_n + a_{n-1} - 6a_{n-2} = 0 \text{ for } n \geq 2 \]
\[ a_0 = -1, a_1 = 8 \]

If \( a_n = cr^n \) with \( c, r \neq 0 \), then we get \( cr^n + cr^{n-1} - 6cr^{n-2} = 0 \).
The characteristic equation is \( r^2 + r - 6 = 0 \). So
\[ 0 = r^2 + r - 6 = (r + 3)(r - 2), \text{ and } r = 2, -3. \]

Since we have two real roots, we get the two solutions \( a_n = 2^n \) and
\[ a_n = (-3)^n \] - these are linearly independent solutions, since neither
is a multiple of the other. The general solution is:
\[ a_n = c_1(2^n) + c_2(-3)^n, \text{ where } c_1, c_2 \text{ are arbitrary constants.} \]

Since \( a_0 = -1, a_1 = 8 \), we get the values for \( c_1, c_2 \) as follows:
\[ -1 = a_0 = c_1(2^0) + c_2(-3)^0 = c_1 + c_2 \]
\[ 8 = a_1 = c_1(2^1) + c_2(-3)^1 = 2c_1 - 3c_2 \]

Solving the system of equations, we get \( c_1 = 1, c_2 = -2 \). So
\[ a_n = 2^n - 2(-3)^n, n \geq 0. \]
This is the unique solution for the given recurrence relation.
Solving Recurrence Relations

**Exercise:** Solve the recurrence relation that produces the Fibonacci numbers:

\[ F_{n+2} = F_{n+1} + F_n \]

for \( n \geq 0 \) and \( F_0 = 0, F_1 = 1 \).
**Example:** For $n \geq 0$, let $S = \{1, 2, 3, \ldots\}$ and let $a_n$ be the number of subsets of $S$ that contain only values less than or equal to $n$, and that contain no consecutive integers. Find and solve a recurrence relation for $a_n$.

For $n = 0$: $a_0 = 1$
For $n = 1$: $a_1 = 2$
For $n = 2$: $a_2 = 3$
For $n = 3$: $a_n = 5$

Assume $n \geq 2$. Consider a subset $A$ of $S$ with elements that are less than or equal to $n$. If $A$ is counted in $a_n$ (that is, it contains no consecutive integers), then there are 2 cases:

Case 1: $n \in A$. In this case $n - 1 \not\in A$. So the number of such subsets is counted in $a_{n-2}$.

Case 2: $n \not\in A$. The number of such subsets is counted in $a_{n-1}$.

So $a_n = a_{n-2} + a_{n-1}$ for $n \geq 2$
$a_0 = 1$
$a_1 = 2$

The characteristic equation is $r^2 - r - 1 = 0$. So the characteristic roots are $r = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$.

**Exercise:** Finish the solution.
Case 3 - Repeated Real Roots

When we solve a second-order linear homogeneous recurrence relation with constant coefficients and get a root of multiplicity 2, we get the following solution.

**Theorem:** If \( b_0a_n + b_1a_{n-1} + b_2a_{n-2} + \ldots + b_ka_{n-k} = 0 \), and \( b_0, b_k \neq 0 \), \( b_0, b_1, \ldots, b_k \) real constants, and \( r \) is a characteristic root of multiplicity \( m \), where \( 2 \leq m \leq k \), then the part of the general solution that involves the root \( r \) has the form
\[
c_0r^n + c_1nr^n + c_2n^2r^n + \ldots + c_{m-1}n^{m-1}r^n,
\]
where \( c_0, \ldots, c_{m-1} \) are constants.

**Example:**
\[
a_{n+2} = 4a_{n+1} - 4a_n, \quad n \geq 0
\]
\[
a_0 = 1, \quad a_1 = 3
\]

The characteristic equation is \( r^2 - 4r - 4 = 0 \), and the characteristic roots are both \( r = 2 \) (ie, 2 is a root of multiplicity 2, since \( r^2 - 4r - 4 = (r - 2)^2 \). So our general solution is

\[
a_n = c_02^n + c_1n2^n.
\]

Use the boundary conditions to determine \( c_0, c_1 \):

\[
1 = a_0 = c_0
\]
\[
3 = a_1 = 2 + 2c_1 \Rightarrow c_1 = 1/2.
\]

So the solution is \( a_n = 2^n + \frac{1}{2}n(2^n) \), or \( a_n = 2^n + n2^{n-1} \) for \( n \geq 0 \).
Exercise: Solve the following recurrence relation.

\[ a_n = 6a_{n-1} - 9a_{n-2} \text{ for } n \geq 2 \]
\[ a_0 = 5, \ a_1 = 12 \]