Stable Matching
(a.k.a. Stable Marriage)

Match N students
to N positions
Each student/position has ranked preference

Students (men)

\[ X > Y \ (A) \]

\[ X > Y \ (B) \]

Medical Schools (women)

\[ X \] \(\bigcirc\) \(A > B\)

\[ Y \] \(\bigcirc\) \(A > B\)

\[ \uparrow \text{unstable} \quad A \& X \text{ will elope} \]

Matching is stable if
No unmatched pair both prefer each other to current partners.

Goal: given all preference lists, find a
 Stable matching.
Example
Y > X > Z (A)  
\( \times \)  A > B > C  
Y > X > Z (B)  
\( \times \)  A > B > C  
X > Y > Z (C)  
\( \times \)  A > B > C

Gale–Shapley:
Start with nobody matched.

Repeat:

Pick any unattached man.
This man proposes to next woman on his list.
The woman:
  - if unattached or preferred to current match, accepts & breaks up current match
  - else, she prefers current match \( \Rightarrow \) reject
  - if rejected, man crosses her off his list.

Claims
(1) This terminates.
- every woman's happiness never decreases.
- in each round, either:
  \( \rightarrow \) man crossed off a man's list
  \( \rightarrow \) woman gets happier by \( \geq 1 \)
\( n \) men \& \( n \) per list \( \Rightarrow \) only \( n^2 \) times
\( n \) women \& \( n \) per list \( \Rightarrow \) only \( n^2 \) times
\( \Rightarrow \) \( O(n^2) \) steps
(2) The final state is stable

To prove otherwise, suppose in the final state
some A would elope with Y, while
A matches to X & Y to B

Y > X  A   X
    |   |
    |   |
    B   Y  A > B

How did this happen?
For A to be matched to X,
A proposed to X
=> A at some point proposed to Y.
=> at some point, Y had a mate
she thought > A in quality

But Y's happiness never decreases
=> Y's final happiness is > her view of A.
which B is no worse.

=> Gale-Shapley gives a stable matching in \( O(n^2) \) time.

Theorem: Gale-Shapley is optimal for men
& optimal for women.

For all stable matchings, A gets best match in any
X worst in any [ ]
Lemma: Each man only rejected by a woman that cannot be matched in any stable matching.

Proof (induct on # rounds)

in a given round, suppose X rejects A in favor of B.
B has been rejected by all he prefers to X.
\[ \Rightarrow \] all of those infeasible for B, by induction
\[ \Rightarrow \] in any stable matching, B would elope with X if X agrees.
\[ \Rightarrow \] in any stable matching, X is not matched to A (would cause elopement)
\[ \Rightarrow \] A infeasible for X.

\[ \Rightarrow \] Theorem.