Linear Programming

Developed in WW2 era, Leonid Kantorovich

N variables, M constraints
Optimize linear function
Subject to linear constraints

\[ 5x_1 + 6x_2 - 3x_3 + x_4 \]

s.t.
\[ x_3 \geq x_1 + 2x_2 + 3 \]
\[ x_4 = x_1 + x_2 \]
\[ x_1 \geq 0 \]
\[ x_2 \leq 1 \]

"Standard form" vs "Symmetric"

\[ \max c^T x \]

s.t.
\[ x \geq 0 \]
\[ Ax \leq b \]

Alternative form:
\[ x_1 + 2x_2 - x_3 \leq -3 \]
\[ x_1 + x_2 - x_4 \geq 0 \]
\[ -x_1 - x_2 + x_4 \leq 0 \]
\[ -x_1 \leq 0 \]
\[ x_2 \leq 1 \]

Equational form:
\[ \max c^T x \]

s.t.
\[ Ax = b \]
\[ x \geq 0 \]
Example:

\[
\begin{align*}
\text{max} & \quad 6x + 3y \\
\text{s.t.} & \quad x \geq 0, \quad y \geq 0 \\
& \quad 2x + 2y \leq 5 \\
& \quad 5x + y \leq 4
\end{align*}
\]

Feasible set

\[\text{direction to maximize} \]

will be maximized at a vertex.

But many possible vertices.

\(n\) unknowns, so \(n\) equations give unique solution (typically)

\[\binom{m+n}{n} \leq (m+n)^n \] vertices.

Simplex method: walk along vertices

Given a vertex, walk edges

Edge corresponds to an equation to relax.

Move along steepest implement direction till another constraint tight. (\(=\) hit another vertex)

Interior point \(\text{polynomial} \): \(n^m\) \text{ was proven in practice depends on problem, open: strongly polynomial}
LP $\Rightarrow$ "does the polytope have any solution?"

- Binary search on $\beta = c^T x$, see $c^T x \geq \beta$ on constraints.

Of course, max flow is on LP.

$\text{Max } \sum f_{uv}$

subject to

- $\sum f_{uv} = 0 \text{ for } u \neq s$,
- $f_{uv} \leq c_{uv}$, $u, v, v, w$,
- $f_{uv} \geq 0$.

Min cut? equivalent minimization problem.

$\text{Min } \sum Y_{uv} c_{uv}$

subject to

- $Y_{uv} - Y_u + Y_v \geq 0$ for $(u, v), u, v \neq s$,
- $Y_{sv} + Y_v \geq 1$ for $(s, u)$,
- $Y_{uv} - Y_u \geq 0$ for $(u, v)$,
- $Y_u + \geq 0$.

$Y_{uv} = \text{"is u-v cut? u \in S, v \notin S"}$

$Y_u = \text{"is u \in S? 1 if true, 0 otherwise"}$

$Y_s, Y_{uv} \in \{0, 1\}$

1. $u \in S \Rightarrow v \in S$ or $uv$ cut,
2. $u \in S \Rightarrow ut$ cut, $s$ cut, $s, u + y_s = \frac{4}{3}$
3. $v \notin S \Rightarrow sv$ cut, $s$ cut, $s, v + y_s = 0$

Not obviously integral.
Duality:

\[ P = \max \ 6x + 3y \]
\[ \text{st. } x \geq 0, \ y \geq 0, \ 2x + 2y \leq 5, \ 5x + y \leq 4, \ x + 3y \leq 3. \]

Any upper bound:
\[ 6x + 3y \leq 6x + 6y \leq 3 \cdot 5 = 15, \]
\[ 6x + 3y \leq 7x + 3y \leq 5 + 4y = 9. \]

Let \( \alpha, \beta \) satisfy
\[ 2\alpha + 5\beta \geq 6, \alpha, \beta \geq 0, \ 2\alpha + \beta \\geq 3, \]
then
\[ 6x + 3y \leq 5\alpha + 4\beta + 3y. \]

\[ D = \min \ 5\alpha + 4\beta + 3y \]
\[ \text{st. } 2\alpha + 5\beta \geq 6, \alpha, \beta \geq 0, \]
\[ 2\alpha + \beta \geq 3, \alpha, \beta \geq 0. \]

**Weak Duality:**
\[ P \leq D \]
\[ (\text{easily shown}) \]

**Strong Duality:**
\[ P = D \]
\[ (\text{harder to prove for LP}) \]

\[ \max c^T x \iff \min b^T y \]
\[ \text{st. } A x \leq b, \ x \geq 0 \]
\[ \min b^T y \]
\[ \text{st. } A^T y \geq c, \ y \geq 0 \]
\[ A^T y = c. \]
Non-negative variables $\iff$ inequality constraints
unconstrained variables $\iff$ equality constraints

\[ x_1 = \# \text{ flat+bed produced/month} \]
\[ x_2 = \# \text{ economy} \]
\[ x_3 = \# \text{ luxury} \]

\[ \frac{1}{2} x_1 + 2 x_2 + x_3 \leq 24 \quad \text{days of metalworking} \]
\[ x_1 + 2 x_2 + 4 x_3 \leq 60 \quad \text{days of wood working} \]
\[ x_1, x_2, x_3 \geq 0 \]

Max \[ z = 6 x_1 + 14 x_2 + 13 x_3 \]

Dual: minimize \[ 24 y_1 + 60 y_2 \]

s.t. \[ y_1, y_2 \geq 0 \]
\[ \frac{1}{2} y_1 + y_2 \geq 6 \]
\[ 2 y_1 + 2 y_2 \geq 14 \]
\[ y_1 + 4 y_3 \geq 13 \]

\[ y_1 = 11, \ y_2 = \frac{1}{2} \implies 294 \]

\[ x_1 = 36, \ x_3 = 6 \implies 216 + 78 = 294 \]

Build 36 flat+beds, 6 luxury

One more day of metalworking is worth \( \frac{1}{2} \)
proof of duality

we know: \( P \leq D \), weak duality

but also: \( P = D \)

let \( x^* = \text{opt for } P \), \( I \) be tight constraints.

then \( c \) lies in cone \( c + x^* \); \( a_i^t x^* = b_i \).

\( \leq a_i^t y \quad y \geq 0 \)

\( \supp(y) \subseteq I \)

\( b^t y = \sum_{i \in I} b_i y_i = \sum_{i \in I} (a_i^t x^*) y_i = \sum_{i \in I} a_i y_i x_i^* = c^t x^* \)

\( = \sum_{i \in I} (a_i^t x_i^*) y_i = \sum_{i \in I} x_i^* \cdot \sum_{j \in I} y_i \)