1. Consider the following variant of interval scheduling. You have \( n \) intervals, each with a given integer start and end time \([s_i, t_i]\) and cost \( c_i \), and would like to choose a subset \( S \) that minimizes the cost

\[
\text{cost}(S) = \sum_{i \in S} c_i
\]

subject to the constraint that every integer time in \([0, T]\) is covered by at least at least \( k \) different intervals in \( S \).

Show how to reduce this problem to a minimum cost circulation problem. You may assume that \( T = O(n) \).

2. [Problem 372 of Brian Dean’s book: the maximum-density subgraph problem.] Given an undirected, unweighted graph, show how to compute the subgraph of maximum edge density using minimum cuts. Hint: for the \( s-t \) graph construction provided in the book, and any cut \( S \), express \(|\text{cut}(S)|\) in terms of the number of edges in \( S \) and the degrees of vertices in \( S \). How does this relate to the edge density in \( S \)?

3. In regression, you are given a set of points \((x_i, y_i)\) and would like to find a line \( y = mx + b \) such that the error is small by some measure. In \( \ell_1 \) regression, one would like to minimize the \( \ell_1 \) norm of the residuals:

\[
\sum_{i=1}^{n} |(\alpha x_i + \beta) - y_i|.
\]

The goal is to find \( \alpha \) and \( \beta \) minimizing this quantity.

(a) Show how to express this problem as a linear program. Hint: the constraint \(|a| \leq b\) is equivalent to the two constraints \( a \leq b \) and \(-a \leq b\).

(b) Write your program in the primal form

Maximize \( c^T x \)

Subject to \( Ax \leq b \)
(c) Give the asymmetric dual form of your linear program.

\[
\text{Minimize } b^T x \\
\text{Subject to } Ax = c
\]

(d) Prove that, in the optimal regression, half the points lie above the line and half the points lie below the line.

(e) Give a direct interpretation of the dual LP, explaining what each expression/variable signifies and why the result is correct. Hint: the dual variables correspond to whether \( y_i \) is above or below the optimal regression line.

You may find it helpful to assume that no points lie on the optimal regression line. You may assume this, though I encourage you to figure out what happens in general.