Problem Set 7

CS 331H

Due Thursday, April 6

1. [Problem 372 of Brian Dean’s book.] Given an undirected, unweighted graph, we would like to compute the subgraph of maximum edge density. The edge density of a subgraph is the number of edges divided by the number of vertices.

Consider the following construction. For a given “guess” $\lambda$, construct a dummy source $s$ and sink $t$. Draw an edge from $s$ to each graph node $u$ of capacity $m$; one from each graph node $u$ to $t$ of capacity $m + 2\lambda - d_u$, where $d_u$ is the degree of $u$ in the original graph; and give each edge $(u, v)$ in the original graph capacity 1.

(a) For a nonempty set $S$ of vertices in the original graph, express the cost of cutting $S \cup \{s\}$ from the rest of the graph, in terms of the number of edges fully contained in $S$ and the degrees in $S$.

(b) Show that this value is less than $mn$ if, and only if, the edge density of $S$ is more than $\lambda$.

(c) Show how a max-flow algorithm and binary search can narrow down on the maximum edge density of any subgraph. Show that after $O(\log n)$ steps of binary search, you can compute the maximum edge density exactly.

(d) Show how to compute the set $S^*$ of maximum edge density, not just its value.

2. Work through the Jupyter notebook on the website.