1 Overview

This lecture is about Markov Chains, a type of stochastic process where the distribution of the process at time $t$ depends only on the value of the process at time $t - 1$.

2 Introduction

**Definition 2.0.1** (Markov Chain). A Markov Chain $(X_t)_{t \in \mathbb{N}}$ is a sequence of random variables on some state space $S$ which obeys the following property:

$$\forall t > 0, (s_i)_{i=0}^t \in S, \mathbb{P} \left[ X_t = s_t \bigg| \bigcap_{i=0}^{t-1} (X_i = s_i) \right] = \mathbb{P} [X_1 = s_t | X_0 = s_{t-1}]$$

We will write these probabilities as a *transition matrix* $P$, where $P_{ij} = \mathbb{P} [X_1 = s_j | X_0 = s_i]$. Note that $\forall i, \sum_j P_{ij} = 1$ is necessary for $P$ to be a valid transition matrix.

If $q \in \mathbb{R}^{|S|}$ is the distribution of $X$ at time 0, the distribution of $X$ at time $t$ will then be $qP^t$.

2.1 Example: Random Walk on a Graph

Let our state space be the vertices of a graph $G = (V,E)$. Then we can define a Markov chain by a random walk on $G$, where at each step the walk jumps to a random neighbour of the current vertex. This gives us the following transition matrix:

$$P_{uv} = \begin{cases} \frac{1}{d(u)} & (u,v) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

3 The Fundamental Theorem of Markov Chains

**Definition 3.0.1** (Ergodicity). A Markov Chain is **ergodic** if $\exists \Pi \in \mathbb{R}^{|S|}$ such that:

$$\forall s \in S, \Pi_s > 0 \\
\lim_{t \to \infty} qP^t = \Pi$$
We will call this $\Pi$ the *stationary distribution* of $X$. Note that when it exists, $\Pi$ is the unique vector $\Pi \in \mathbb{R}^{|S|}$ such that $\Pi P = \Pi$, with $\sum_{s \in S} \Pi_s = 1$ and $\Pi_s \in [0, 1]$ for all $s$.

**Theorem 3.0.2** (The Fundamental Theorem of Markov Chains). Let $X$ be a Markov Chain on a finite state space $S = [n]$ satisfying the following conditions:

1. **Irreducibility** There is a path between any two states which will be followed with $> 0$ probability, i.e. $\forall i, j \in [n], \exists t \mathbb{P}[X_t = j | X_0 = i] > 0$.

2. **Aperiodicity** Let the period of a pair of states $u, v$ be the GCD of the length of all paths between them in the Markov chain, i.e. $\gcd\{t \in \mathbb{N} > 0 | \mathbb{P}[X_t = v | X_0 = u] > 0\}$. $X$ is aperiodic if this is $1$ for all $u, v$.

Then $X$ is ergodic.

Note that both these conditions are necessary as well as sufficient.

### 3.1 Further Definitions

$$N(i, t) = |\{t \in \mathbb{N} | X_t = i\}|$$

This obeys $\lim_{t \to \infty} \frac{N(i, t)}{t} = \Pi_i$ for an ergodic chain with stationary distribution $\Pi$.

$$h_{u,v} = \mathbb{E}[\min\{t | X_t = v\} | X_0 = u]$$

This is called the hitting time of $v$ from $u$, and it obeys $h_{i,i} = \frac{1}{\Pi_i}$ for an ergodic chain with stationary distribution $\Pi$.

### 4 Random Walks on Undirected Graphs

We consider a random walk $X$ on a graph $G$ as before, but now with the assumption that $G$ is undirected.

Clearly, $X$ will be irreducible iff $G$ is connected. It can also be shown that it will be aperiodic iff $G$ is not bipartite. The $\Rightarrow$ direction follows from the fact that paths between two sides of a bipartite graph are always of even length, whereas the $\Leftarrow$ direction follows from the fact that a non-bipartite graph always contains a cycle of odd length.

We can always make a walk on a connected graph ergodic simply by adding self-loops to one or more of the vertices.

#### 4.1 Ergodic Random Walks on Undirected Graphs

**Theorem 4.1.1.** If the random walk $X$ on $G$ is ergodic, then its stationary distribution $\Pi$ is given by $\forall v \in V, \Pi_v = \frac{d(v)}{2m}$.
Proof. Let $\Pi$ be as defined above. Then:

\[(\Pi P)_v = \sum_{u,v \in E} \Pi_u \frac{1}{d(u)} = \sum_{u,u,v \in E} \frac{1}{2m} = \frac{d(v)}{2m} = \Pi_v\]

So as $\sum_v \Pi_v = \frac{2m}{2m} = 1$, $\Pi$ is the stationary distribution of $X$. \qed

In general, even on this subset of random walks, the hitting time will not be symmetric, as will be shown in our next example. So we define the commute time $C_{u,v} = h_{u,v} + h_{v,u}$.

### 4.2 Example: The Lollipop Graph

The lollipop graph on $n$ vertices is a clique of $\frac{n}{2}$ vertices connected to a path of $\frac{n}{2}$ vertices. Let $u$ be any vertex in the clique that does not neighbour a vertex in the path, and $v$ be the vertex at the end of the path that does not neighbour the clique. Then $h_{u,v} = \theta(n^3)$ while $h_{v,u} = \theta(n^2)$. This is because it takes $\theta(n)$ time to go from one vertex in the clique to another, and $\theta(n^2)$ time to successfully proceed up the path, but when travelling from $u$ to $v$ the walk will fall back into the clique $\theta(1)$ times as often as it makes it a step along the path to the right, adding an extra factor of $n$ to the hitting time.

### 5 Electrical Resistance and Commute Time of a Graph

View graph $G$ as an electrical network with unit resistors as edges. Let $R_{u,v}$ be the effective resistance between vertices $u$ and $v$. The commute time between $u$ and $v$ in a graph is related to $R_{u,v}$ by $C_{u,v} = 2mR_{u,v}$. We get the following inequalities assuming this relation.

If $(u,v) \in E$,

\[R_{u,v} \leq 1 : C_{u,v} \leq 2m\]

In general, $\forall u,v \in V$,

\[R_{u,v} \leq n - 1 : C_{u,v} \leq 2m(n - 1) < n^3\]
We inject \( d(v) \) amperes of current into \( \forall v \in V \). Subsequently we pick some vertex \( u \in V \) and remove \( 2m \) current from \( u \) leaving net \( d(u) - 2m \) current at \( u \). Now we get voltages \( x_v \) \( \forall v \in V \). Assume we have \( x_v - x_u = h_{v,u} \) \( \forall v \neq u \in V \) (will prove subsequently). Let \( L \) be the Laplacian for \( G \) and \( D \) be the degree vector, then we have

\[
Lx = i_u = D - 2m \mathbb{1}_u
\]

\[
\forall v \in V, \sum_{(u,v) \in E} x_v - x_u = d(v)
\] (1)

5.1 Lollipop Graph

Let us revisit the lollipop graph with the electrical network view and compute \( h_{u,v} \) and \( h_{v,u} \) with \( u \) and \( v \) as before. To compute \( h_{u,v} \). Let \( u' \) be the vertex common to the clique and the path. Clearly, the path has resistance \( \theta(n) \). \( \theta(n) \) current is injected in the path and \( \theta(n^2) \) current is injected in the clique.

Consider draining current from \( v \). The current in the path is \( \theta(n^2) \) as \( 2m - 1 = \theta(n^2) \) current is drained from \( v \) which enters \( v \) through the path implying \( x'_u - x_v = \theta(n^3) \) using Ohm’s law \( (V = IR) \). Now consider draining current from \( u \) instead. The current in the path is now \( \theta(n) \) implying \( x_v - x'_u = \theta(n^2) \) by the same argument.

Since the effective resistance between any edge in the clique is less than 1 and \( \theta(n^2) \) current is injected, there can be only \( \theta(n^2) \) voltage gap between any 2 vertices in the clique. We get \( h_{u,v} = x_u - x_v = \theta(n^3) \) in the former case and \( h_{v,u} = x_v - x_u = \theta(n^2) \) in the latter.

5.2 Proof of Relation

Define \( h'_{v,u} = h_{v,u} \) when \( v \neq u \) except \( h'_{v,v} = 0 \). By current conversion, \( \forall u \neq v \in V \), we have

\[
h'_{v,u} = \sum_{(v,w) \in E} \frac{1}{d(v)} (1 + h'_{w,u})
\]

\[
h'_{v,u} = 1 + \sum_{(v,w) \in E} \frac{1}{d(v)} h'_{w,u}
\]

\[
d(v) = \sum_{(v,u) \in E} h'_{v,u} - h'_{w,u}
\] (2)

Equations 1 and 2 are linear systems with unique solutions and are identical under \( x_v - x_u = h'_{v,u} \) (up to same additive shift to each entry). \( x_v = h'_{v,u} \) if \( x_u = 0 \).

We have shown that for \( i_u = D - 2m \mathbb{1}_u \) with \( x = L^+ i_u \) that \( x_v - x_u = h_{v,u} \). For \( u' \), we have \( x' = L^+ i_{u'} \). Now, we have,

\[
x - x' = L^+(i_u - i_{u'}) = 2mL^+(e_{u'} - e_u)
\]
The above is equivalent to $2m$ times voltage obtained if you inject 1 ampere at $u'$ and remove 1 ampere from $u$. Using Kirchoff’s law we get

$$2mR_{u,u'} = (x - x')_{u'} - (x - x')_u$$
$$= (x_{u'} - x_u) - (x'_u - x'_{u'})$$
$$= h_{u',u} + h_{u,u'} = C_{u,u'}$$

6 Cover Time of a Graph

We define $C_u(G)$ as the expected time for a random walk starting at $u$ to visit all vertices in a graph. $C(G)$ is the maximum of $C_u(G)$ over all $u \in V$.

We have $\forall u \in V$,

$$C_u(G) \leq 2m(n - 1)$$

Consider the spanning tree $T$ of graph $G$. The cover time is bounded by traversing the edges of the tree in both directions (as we could just do a DFS on the spanning tree), and hitting time gives the expected time of moving along an edge, we get

$$C_u(G) \leq \sum_{(u,v) \in E(T)} h_{u,v} + h_{v,u}$$
$$= \sum_{(u,v) \in E(T)} C_{u,v}$$
$$\leq (n - 1) \max_u C_{u,v}$$
$$\leq 2m(n - 1)$$

This above inequality is tight for lollipop ($\theta(n^3)$) but not for cliques which has $O(n \log n)$ as we can model it as a coupon collector problem.

Let $R_{max} = \max_{u,v \in V} R_{u,v}$. We give a tighter bound without proof on $C(G)$ as follows:

$$mR_{max} \leq C(G) \lesssim mR_{max} \log n$$

References