1 Balls and Bins Revisit

Question:

1. How many balls can I throw in, typically, into \( n \) bins before

(a) max load is 2 with 1 choice?

(b) max load is 2 with 2 choices?

2. How many balls can I throw in, typically, into \( n \) bins before

(a) max load is 3 with 1 choice?

(b) max load is 3 with 2 choices?

(1.a) After \( k \) balls are thrown, if max load is 1, then the loads of \( k \) bins are 1 and \( n - k \) bins are 0. Define failure as the case that the max load is 2. The chance of failure of \( k \)-th is \( \frac{k}{n} \).

Between \( \frac{k}{2} \)-th and \( k \)-th balls thrown, the failure prob is \( O \left( \frac{k}{n} \right) \). The total failure probability after \( k \) balls are thrown is approximately \( 1 - \left(1 - \frac{k}{n}\right)^k = \Theta \left( \frac{k^2}{n} \right) \).

(1.b) Same analysis as (1.a). Except now the failure probability of \( k \)-th is \( O \left( \frac{k^2}{n^2} \right) \). So the total failure probability after \( k \) balls are thrown is \( 1 - \left(1 - \frac{k^2}{n^2}\right)^k = \Theta \left( \frac{k^3}{n^2} \right) \).

(2.a) After \( k \) balls \((k < n)\) are thrown,

- expect \( \Theta (k) \) bins with 1 ball.
- expect \( \Theta \left( \frac{k^2}{n} \times k \right) = \Theta \left( \frac{k^3}{n} \right) \) bins with 2 ball.
- with \( \Theta \left( \frac{k^3}{n^2} \right) \) probability that there exists a bin with height 3.
Typically, we can throw $\Theta(n^{2/3})$ balls before the max load is 3.

(2.b) After $k$ balls ($k < n$) are thrown,

- expect $\Theta(k)$ bins with 1 ball.
- expect $\Theta\left(\frac{k}{n} \times \frac{k}{n} \times k\right) = \Theta\left(\frac{k^3}{n^2}\right)$ bins with 2 ball.
- with $\Theta\left(\frac{k^7}{n^6}\right)$ probability that there exists a bin with height 3.

Typically, we can throw $\Theta(n^{6/7})$ balls before the max load is 3.

**Conclusion:** In general, for any height $C = O(1)$. We can throw $k$ balls, typically, before the max load is $C$ where

(a) $k = n^{1 - \frac{1}{C}}$ with 1 choice

(b) $k = n^{1 - \frac{1}{2^{x-1}}}$ with 2 choices

## 2 Random Walk

**Question:** Start at 0. At each step, move +1 or -1 with equal probability. Stop if reach -10 or 100

(a) $\Pr[\text{stop at -10}]$

(b) How long does it stop in expectation?

**Solution:**

(a)

**Method 1:** Using expectation

Let $X_t$ denote the distribution of its location after moving $t$ steps. $\mathbb{E}[X_t - X_{t-1}] = \frac{1}{2} \times (+1) + \frac{1}{2} \times (-1) = 0$. Therefore, $\mathbb{E}[X_t]$ remains the same for different $t$. $\mathbb{P}[\text{stops at } -10] \times (-10) + (1 - \mathbb{P}[\text{stops at 100}]) \times 100 = 0$ which indicates $\mathbb{P}[\text{stops at } -10] = \frac{10}{11}$.

**Method 2:** Let $p_i$ denote the probability of stopping at -10 when starting at $i$. We have

$$p_i = \begin{cases} 1 & \text{if } i = -10; \\ 0 & \text{if } i = 100; \\ \frac{1}{2} (p_{i-1} + p_{i+1}) & \text{otherwise.} \end{cases}$$

By solving this equation, We have $p_i$ is linear with $i$ and $p_0 = \frac{10}{11}$. 
Let $n_i$ denote the expected number of steps to stop when starting at $i$. We have
\[
  n_i = \begin{cases} 
    0 & \text{if } i = -10 \text{ or } 100; \\
    \frac{1}{2} (n_{i-1} + 1) + \frac{1}{2} (n_{i+1} + 1) & \text{otherwise}. 
  \end{cases}
\]
By solving this equation, we have $n_i = -i^2 + 90i + 1000$ and $n_0 = 1000$. 

(b)