Problem Set 5
Randomized Algorithms
Due Tuesday, November 14

1. Let $X_1, \ldots, X_n \sim N(0, 1)$ for some $n \geq 10$, and let $Z = \max_i X_i$.

(a) Show that
\[
\frac{1}{2} \leq \frac{\mathbb{E}[Z]}{\sqrt{\log n}} \leq 3
\]

(b) Show that
\[
\frac{\mathbb{E}[Z]}{\sqrt{2 \log n}} = 1 - o(1)
\]
as $n \to \infty$.

2. Let $X_1$ and $X_2$ be zero-mean subgaussians with parameters $\sigma_1$ and $\sigma_2$, respectively.

(a) Show that if $X_1$ and $X_2$ are independent, then $X_1X_2$ is subgamma. What are the parameters in terms of $\sigma_1$ and $\sigma_2$?

(b) Show that in general, without assuming independence, $X_1 + X_2$ is subgaussians with parameter $2\sqrt{\sigma_1^2 + \sigma_2^2}$.

3. Suppose you can sample elements $x_1, \ldots, x_m$ from a mean zero subgaussian variable $X$, and would like to compute its kurtosis
\[
\kappa = \frac{\mathbb{E}[X^4]}{\mathbb{E}[X^2]^2}.
\]

(a) Consider the algorithm that computes the empirical fourth and second moments $\frac{\sum x_i^4}{m}$ and $\frac{\sum x_i^2}{m}$, and outputs the former divided by the latter squared. How large must $m$ be for this to be an $(\epsilon, \delta)$ approximation to $\kappa$?
(b) Construct a different algorithm that achieves an $(\epsilon, \delta)$ approximation to $\kappa$ with fewer measurements.

4. For a hash family $\mathcal{H}$ from $[U]$ to $[n]$, and a set of items $S \subseteq [U]$, let $X(\mathcal{H}, S)$ be the random variable denoting the load in the first bin:

$$X := |\{i \in S \mid h(i) = 1\}|$$

as a distribution over $h \in \mathcal{H}$. Further, let $f(\mathcal{H}, S)$ denote the expected max load in any bin:

$$f(\mathcal{H}, S) := \mathbb{E}_{h \in \mathcal{H}} \max_{j \in [n]} |\{i \in S \mid h(i) = j\}|.$$  

(a) For any $t \geq 1$, and for any $k$-wise independent hash family $\mathcal{H}$ with $k = O(1)$, and any set $S$ with $|S| = n$, show that

$$\text{Pr}[X \geq t] \lesssim 1/t^k.$$  

**Hint:** bound $\mathbb{E}[X^k]$.

(b) Show that for a $k$-wise independent family $\mathcal{H}$, $k = O(1)$, that

$$f(\mathcal{H}, S) \lesssim n^{1/k}$$

for any $S$ with $|S| = n$.

(c) Show that there exists a pairwise independent hash family $\mathcal{H}$ and set $S$ with $|S| = n$ such that

$$f(\mathcal{H}, S) \gtrsim \sqrt{n}.$$  

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