Problem Set 3

Sublinear Algorithms

Due Tuesday, October 28

1. Recall that $M(X, d, \epsilon)$ denotes the packing number for space $X$ with distance $d$ and radius $\epsilon$, and $N(X, d, \epsilon)$ denotes the covering number. Prove that

$$M(X, d, 2\epsilon) \leq N(X, d, \epsilon) \leq M(X, d, \epsilon)$$

2. Give an algorithm to construct the $k$-tree-sparse approximation of a vector. The input is an integer $k$ and a complete $n$-vertex binary tree $T$ with a nonnegative value $x_v$ associated with each vertex $v$. The output is the set $S$ of size $k$ that includes the root, is a connected subset of the tree, and maximizes $\sum_{v \in S} x_v$.

   (a) Show a simple DP algorithm to solve this in $O(nk^2)$ time and $O(nk)$ space.

   (b) Show how to optimize the algorithm to take $O(nk)$ time.

3. In this problem we show that matrices that satisfy the RIP-2 cannot be very sparse. Let $A \in \mathbb{R}^{m \times n}$ satisfy the $(k, 1/2)$ RIP for $m < n$. Suppose that the average column sparsity of $A$ is $d$, i.e. $A$ has $nd$ nonzero entries.

   Furthermore, suppose that $A \in \{0, \pm \alpha\}^{m \times n}$ for some parameter $\alpha$.

   (a) By looking at the sparsest column, give a bound for $\alpha$ in terms of $d$.

   (b) By looking at the densest row, give a bound for $\alpha$ in terms of $n, m, d$ and $k$.

   (c) Conclude that $d \gtrsim k$. (Recall that this means: there exists a constant $C$ for which $d \geq k/C$.)
(d) What if each non-zero $A_{i,j}$ were drawn from $N(0,1)$?
(e) [Optional] Extend the result to general settings of the non-zero $A_{i,j}$.

4. In class we have shown various algorithms for sparse recovery that tolerate noise and use $O(k \log(n/k))$ measurements, and shown that any $\ell_1/\ell_1$ sparse recovery algorithm must use this many measurements. But what if we don’t care about tolerating noise, and only want to recover $x$ from $Ax$ when $x$ is exactly $k$-sparse?

Consider the matrix

$$
A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{2k-1} & \alpha_2^{2k-1} & \cdots & \alpha_n^{2k-1}
\end{pmatrix}
$$

for distinct $\alpha_i$.

(a) Prove that any $2k \times 2k$ submatrix of $A$ is invertible.
(b) Give an $n^{O(k)}$ time algorithm to recover $x$ from $Ax$ under the assumption that $x$ is $k$-sparse.
(c) [Optional] Give an $n^{O(1)}$ time algorithm to recover $x$ from $Ax$ under the assumption that $x$ is $k$-sparse. You may choose specific values for the $\alpha_i$. Hint: look up syndrome decoding of Reed-Solomon codes.

5. In order to show that SSMP makes progress in each stage, we used a lemma that we will show in this problem.

Let $x_1, \ldots, x_k \in \mathbb{R}^d$, and suppose that

$$
\sum_{i=1}^k \|x_i\|_1 \leq (1 + \delta) \| \sum_{i=1}^k x_i \|_1
$$

for some small enough $\delta$ (say, $\delta = 1/10$). In some sense, this is saying that there is not much “slack” in they are lined up head-to-tail.
(a) Let \( z = \sum_{i=1}^{k} x_i \). Show that there exists an \( i \) such that \( \|z - x_i\|_1 \leq (1 - \frac{\Omega(1)}{k})\|z\|_1 \).

(b) Now suppose \( z = \sum_{i=1}^{k} x_i + w \) for some \( w \in \mathbb{R}^d \) with \( \|w\|_1 \leq \epsilon \|z\|_1 \) for small enough constant \( \epsilon \). Again, show that there exists an \( i \) such that \( \|z - x_i\|_1 \leq (1 - \frac{\Omega(1)}{k})\|z\|_1 \).