Problem Set 3

Sublinear Algorithms

Due Thursday, October 20

1. Recall that $M(X, d, \epsilon)$ denotes the packing number for space $X$ with
distance $d$ and radius $\epsilon$, and $N(X, d, \epsilon)$ denotes the covering number.
Prove that

$$M(X, d, 2\epsilon) \leq N(X, d, \epsilon) \leq M(X, d, \epsilon)$$

2. In this problem we show that matrices that satisfy the RIP-2 cannot
be very sparse. Let $A \in \mathbb{R}^{m \times n}$ satisfy the $(k, 1/2)$ RIP for $m < n$.
Suppose that the average column sparsity of $A$ is $d$, i.e. $A$ has $nd$
nonzero entries.
Furthermore, suppose that $A \in \{0, \pm \alpha\}^{m \times n}$ for some parameter $\alpha$.

(a) By looking at the sparsest column, give a bound for $\alpha$ in terms of $d$.

(b) By looking at the densest row, give a bound for $\alpha$ in terms of $n, m, d$ and $k$.

(c) Conclude that either $d \gtrsim k$ or $m \gtrsim n$. (Recall that this means:
there exists a constant $C$ for which $d \geq k/C$.)

(d) What if each non-zero $A_{i,j}$ were drawn from $N(0,1)$?

(e) [Optional] Extend the result to general settings of the non-zero
$A_{i,j}$.

3. In class we have shown various algorithms for sparse recovery that
tolerate noise and use $O(k \log(n/k))$ measurements, and shown that
any $\ell_1/\ell_1$ sparse recovery algorithm must use this many measurements.
But what if we don’t care about tolerating noise, and only want to
recover $x$ from $Ax$ when $x$ is exactly $k$-sparse?
Consider the matrix

\[ A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{2k-1} & \alpha_2^{2k-1} & \cdots & \alpha_n^{2k-1}
\end{pmatrix} \]

for distinct \( \alpha_i \).

(a) Prove that any \( 2k \times 2k \) submatrix of \( A \) is invertible.

(b) Give an \( n^{O(k)} \) time algorithm to recover \( x \) from \( Ax \) under the assumption that \( x \) is \( k \)-sparse.

(c) [Optional] Give an \( n^{O(1)} \) time algorithm to recover \( x \) from \( Ax \) under the assumption that \( x \) is \( k \)-sparse. You may choose specific values for the \( \alpha_i \). Hint: look up syndrome decoding of Reed-Solomon codes.