Adaptive Sparse Recovery

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Joint work with Piotr Indyk and David Woodruff, 2011-2012

Outline

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1. Motivating Example
2. Formal Introduction to Sparse Recovery/Compressive Sensing
3. Algorithm

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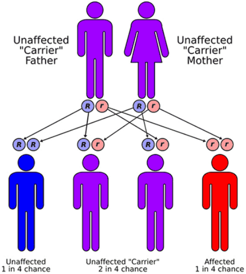
Carrier screening

- Want to figure out who carries a genetic mutation.



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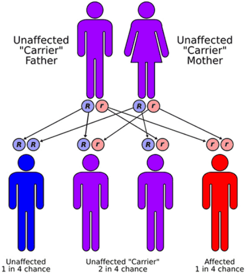

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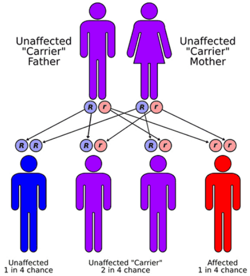

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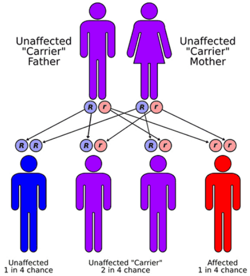

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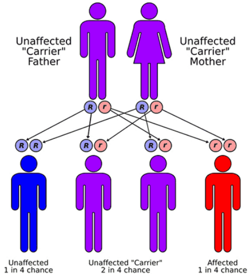

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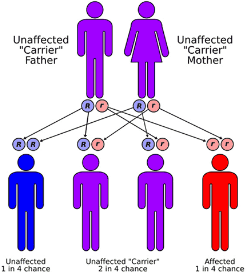

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- Idea: mix together samples.

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- Want to minimize number of tests.

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- **This talk:** $O(k \log\log(n/k))$ adaptive linear measurements.

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- Can we use structure to save on *observations*?

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  - What structure? Sparsity.

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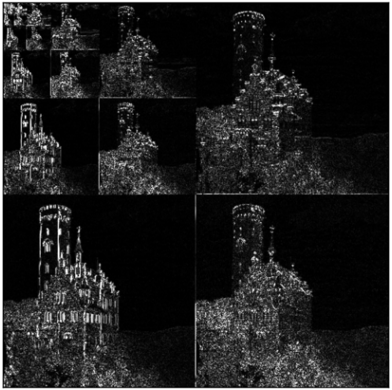

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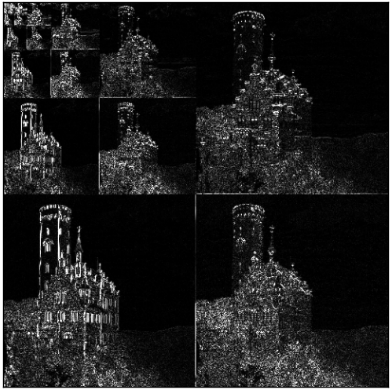

- Same kind of structure as in genetic testing!

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- Note: impossible without using sparsity (A is underdetermined).

Standard Sparse Recovery Framework

- Specify distribution on $m \times n$ matrices A (independent of x).

- Given linear sketch Ax, recover $\hat{x}$.
- Satisfying the recovery guarantee:

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Adaptive Sparse Recovery Framework

- For $i = 1 \ldots r$:
  - Choose matrix A_i based on previous observations (possibly randomized).
  - Observe $A_i x$.
  - Number of measurements m is total number of rows in all A_i.
  - Number of *rounds* is r.
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  - Separating dependence on n and ϵ.

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  - Mapreduce finding most frequent URLs: **Yes**.

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- Lemma implies theorem using standard tricks ([GLPS10]).

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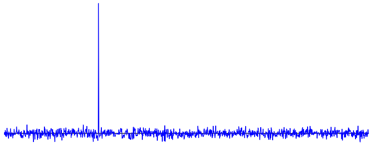

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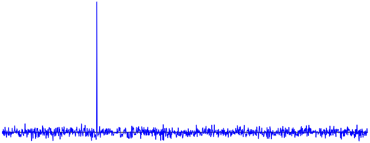

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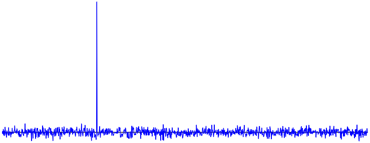

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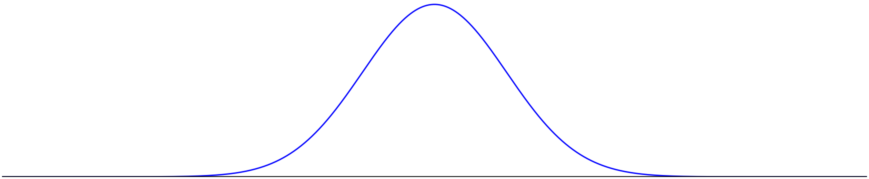

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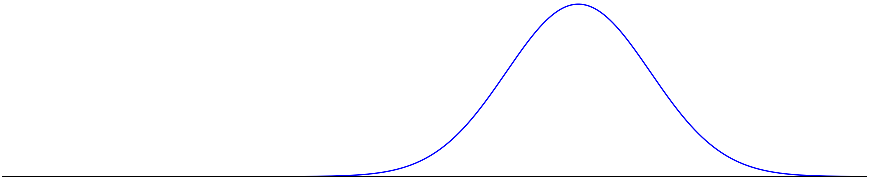

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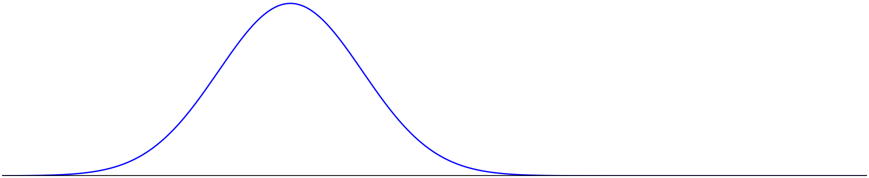

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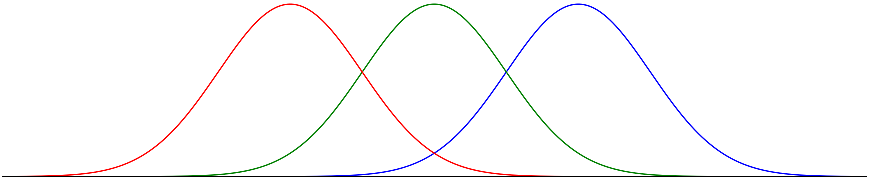

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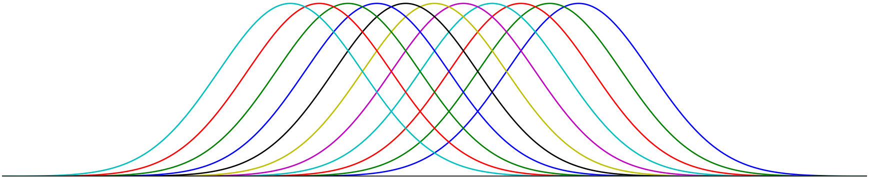

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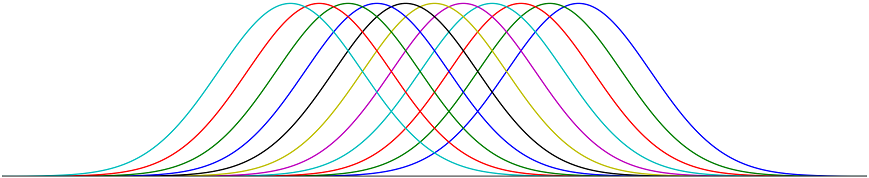

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$$I(i, \langle v, x\rangle) \leqslant \frac{1}{2}\log(1 + \mathrm{SNR})$$

where *SNR* denotes the "signal-to-noise ratio,"

$$SNR = \frac{\mathbb{E}[\text{signal}^2]}{\mathbb{E}[\text{noise}^2]} \lesssim \frac{\mathbb{E}[v_i^2]}{\|v\|_2^2/n} = 1$$

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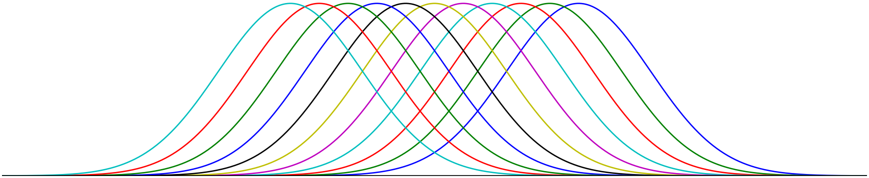

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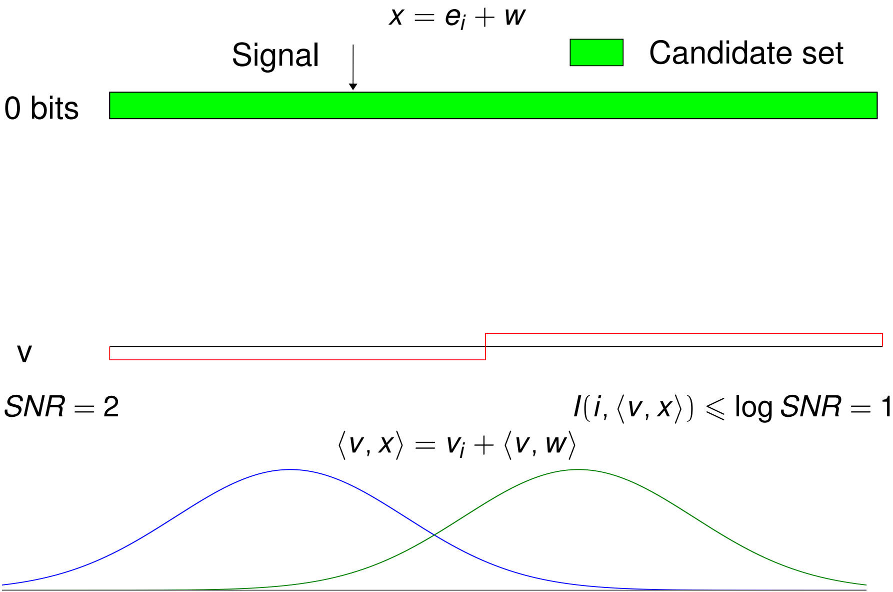

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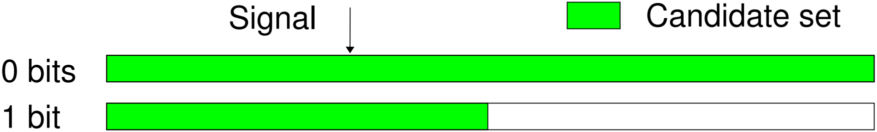
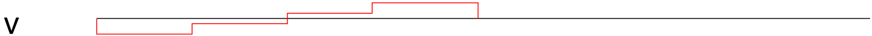
$$x = e_i + w$$





$SNR = 2^2$ $\qquad$ $I(i, \langle v, x \rangle) \leqslant \log SNR = 2$

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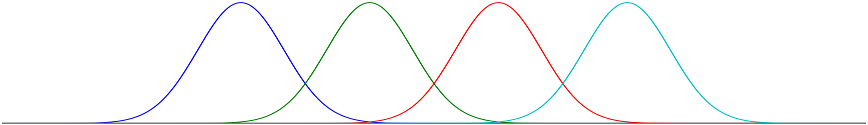

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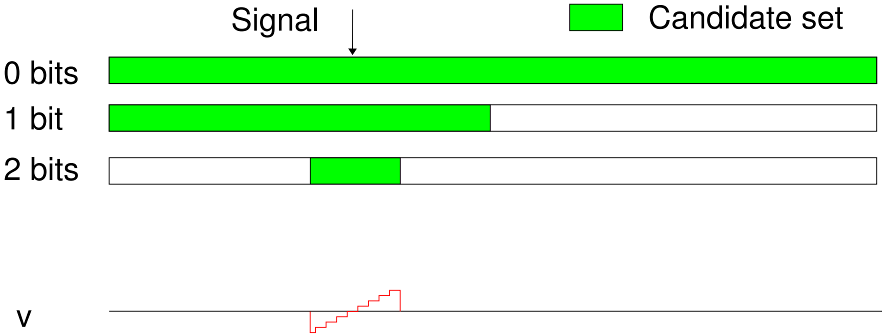

$SNR = 2^4$ $\qquad$ $I(i, \langle v, x\rangle) \leqslant \log SNR = 4$

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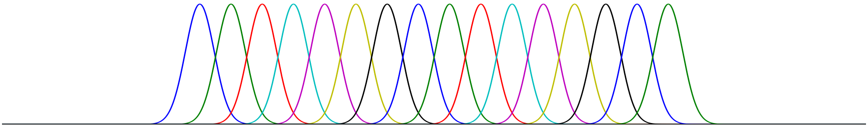

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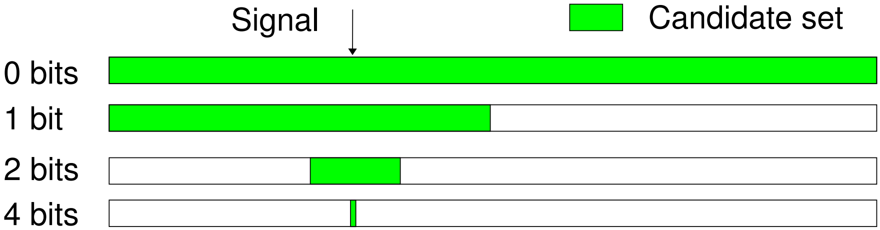

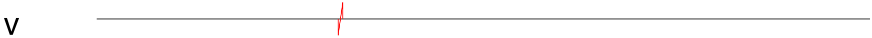

$SNR = 2^8$ $\qquad$ $I(i, \langle v, x \rangle) \leqslant \log SNR = 8$

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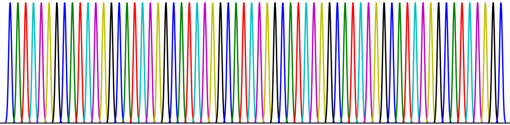

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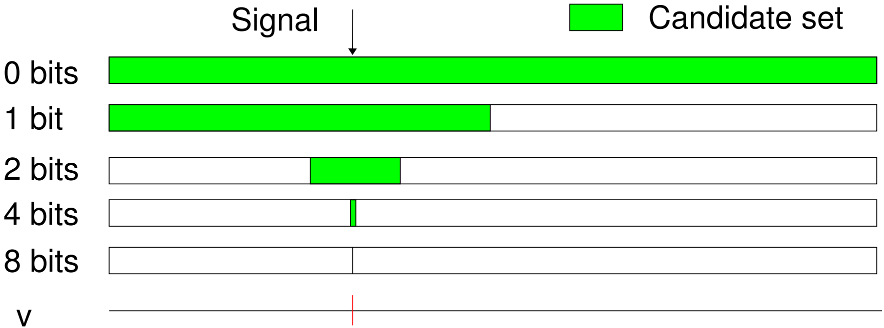

$SNR = 2^{16}$ $\qquad I(i, \langle v, x \rangle) \leqslant \log SNR = 16$

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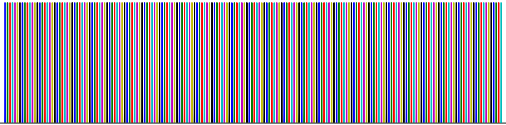

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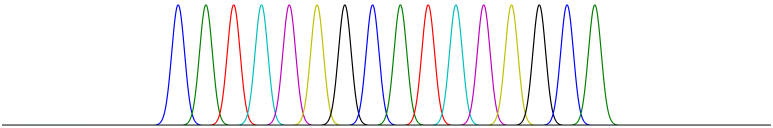
- Once $SNR(x)$ reaches $O(n^2)$, will have $S = \{i\}$.

Recovery when $SNR > n^2$

Getting $\log n$ bits when SNR is n^2



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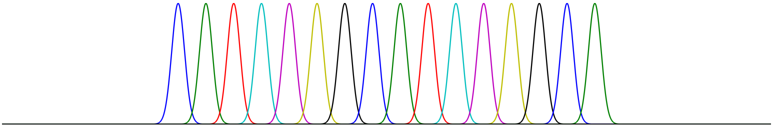

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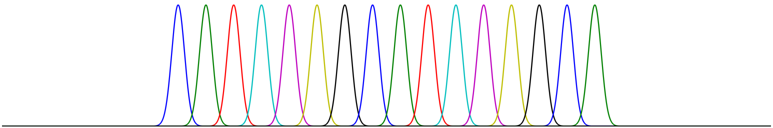

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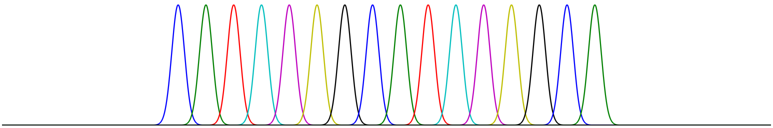

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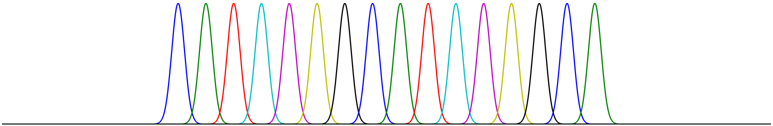

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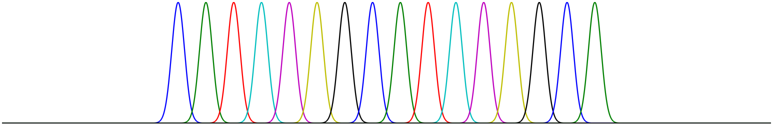

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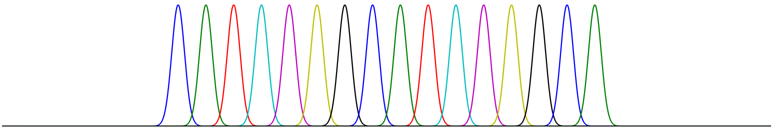

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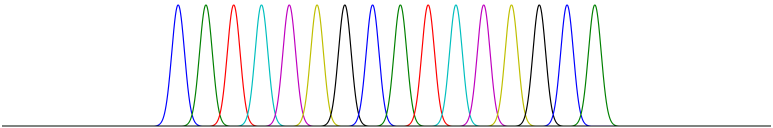

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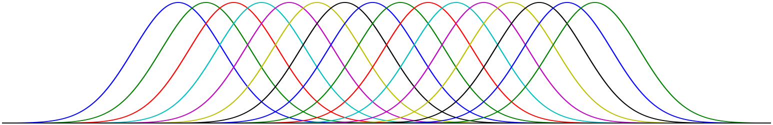
- For $SNR > (n/\delta)^2$, done.

Getting log SNR bits for general SNR



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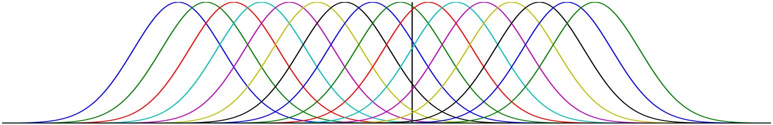
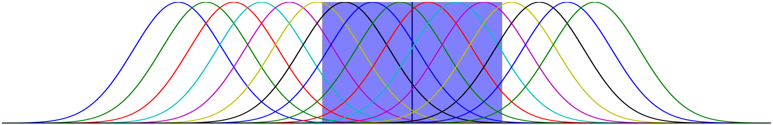


Getting log *SNR* bits for general *SNR*



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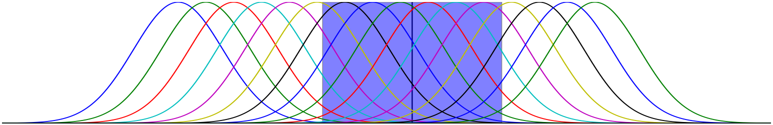

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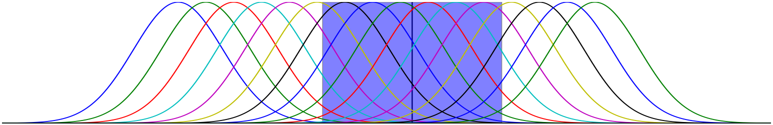

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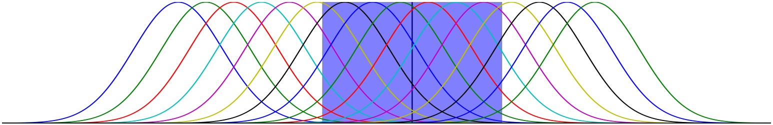

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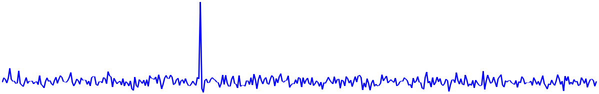

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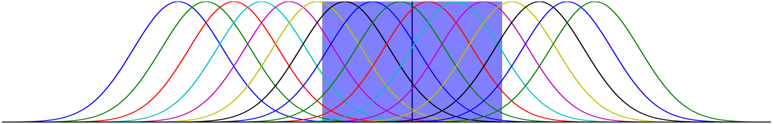

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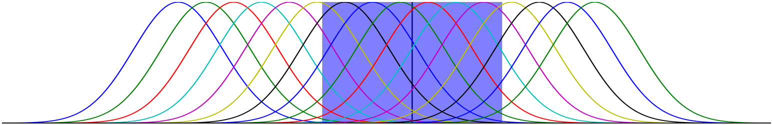

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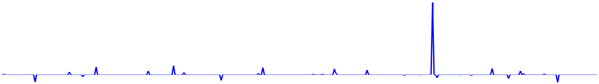

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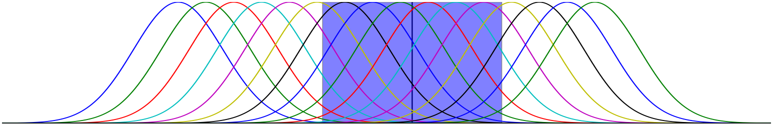

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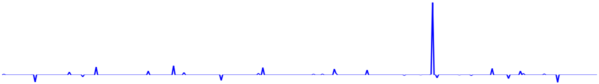

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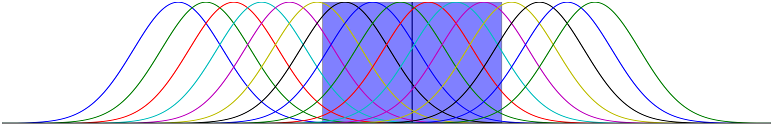

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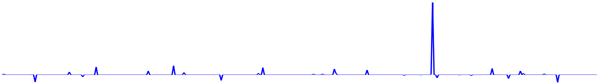

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- Set $\delta = 0.1/2^r$ in round r; still doubly exponential growth.

End proof of key lemma

Lemma

Adaptive C-approximate 1-sparse recovery is possible with $O(\log\log n)$ measurements for some $C = O(1)$.

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- Lemma implies theorem using standard tricks (a la [GLPS10]):

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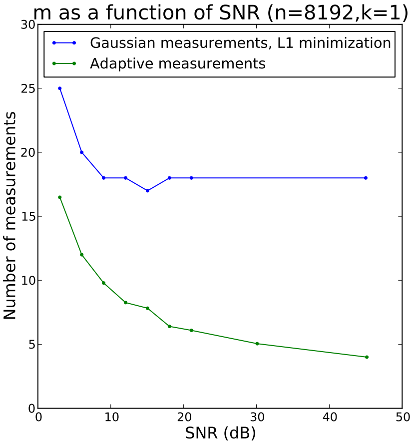

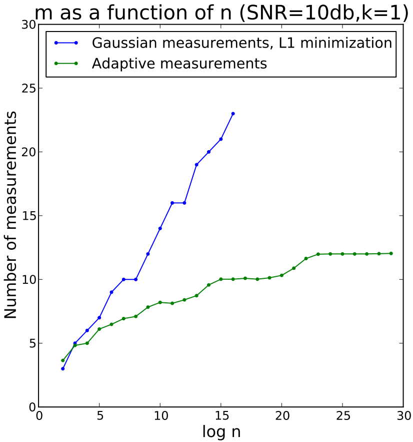

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