Fast RIP matrices with fewer rows

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Princeton MIT Michigan

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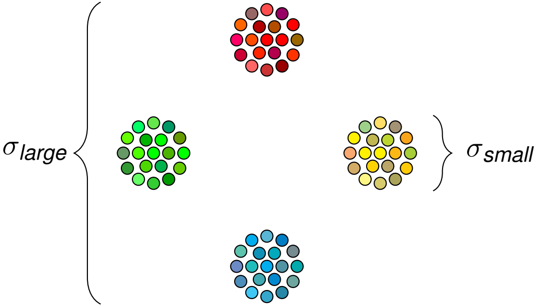

Outline

1 Introduction

- Compressive sensing
- Johnson Lindenstrauss Transforms
- Our result

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    - Overview
    - Symmetrization
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3. Proof
   - Overview
   - Covering Number
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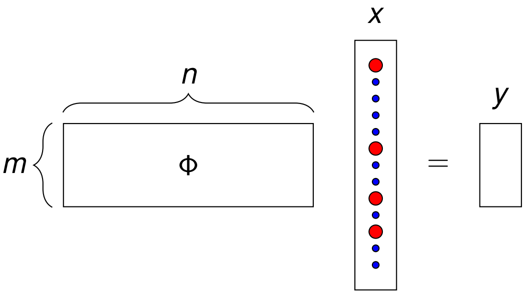
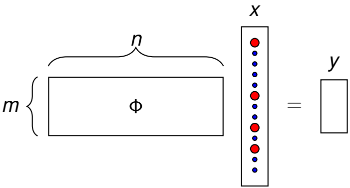
Compressive Sensing

Given: A few linear measurements of an (approximately) k-sparse vector $x \in \mathbb{R}^n$.
Goal: Recover x (approximately).



Compressive Sensing Algorithms: Two Classes



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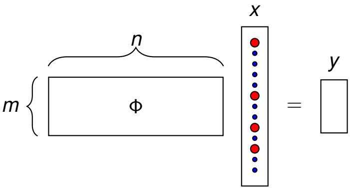

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tied to matrix structure
(e.g. Count-Sketch)

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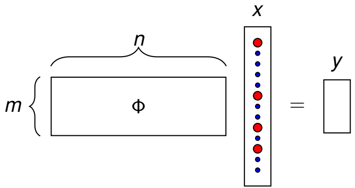

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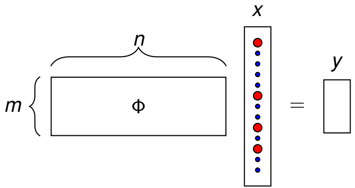

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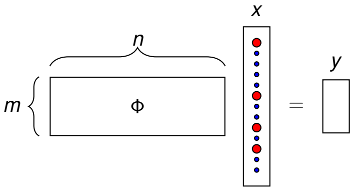

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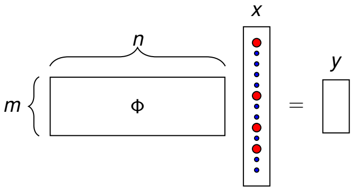

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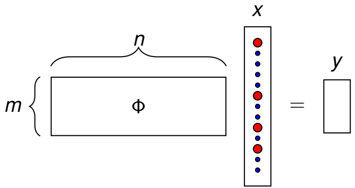

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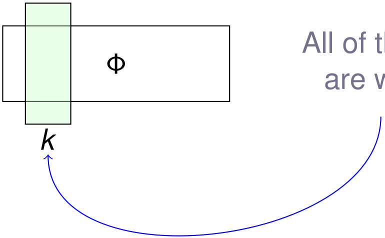
  - the *Restricted Isometry Property* is a sufficient condition.

Restricted Isometry Property (RIP)



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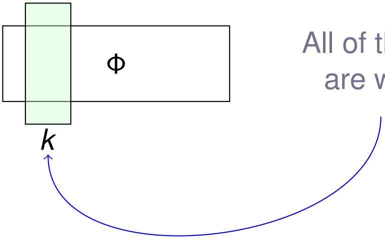

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$$(1-\epsilon)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\epsilon)\|x\|_2^2$$

for all k-sparse $x \in \mathbb{R}^n$.

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  - Random Gaussian matrix: $\Theta(nk \log n)$ time.
- Goal: an RIP matrix with $O(n \log n)$ multiplication and small m.

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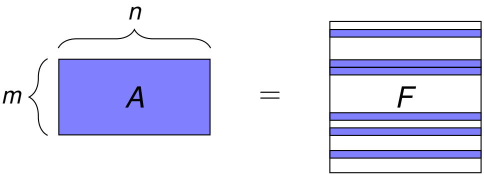
An open question



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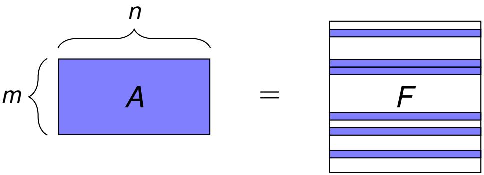

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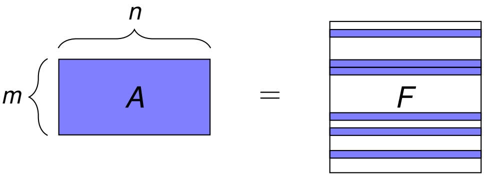

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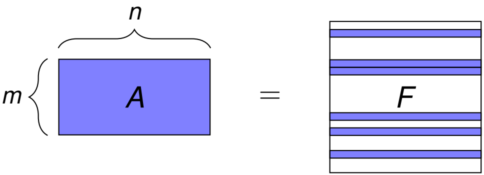

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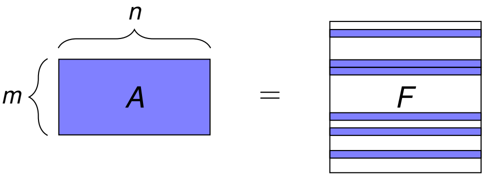

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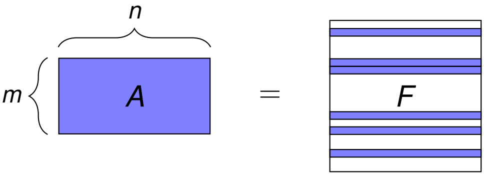

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(Related: how about partial circulant matrices?)

- $m = O(k \log^4 n)$ [RRT12,KMR12].

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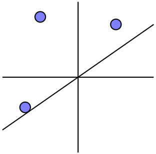
Another motivation: Johnson Lindenstrauss (JL) Transforms



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$S \subset \mathbb{R}^n$

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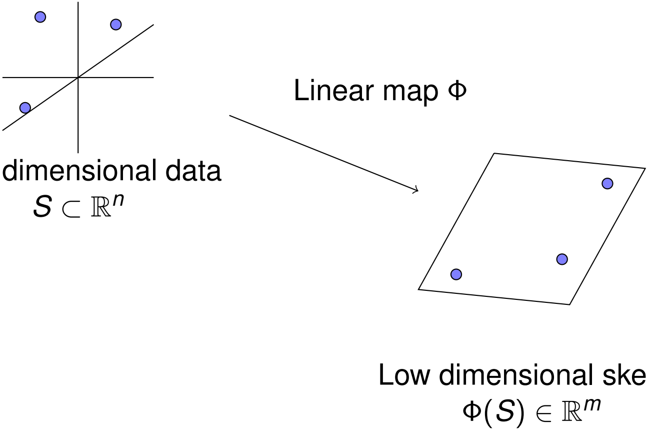

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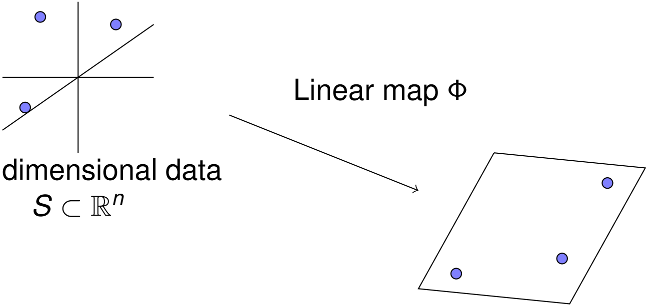

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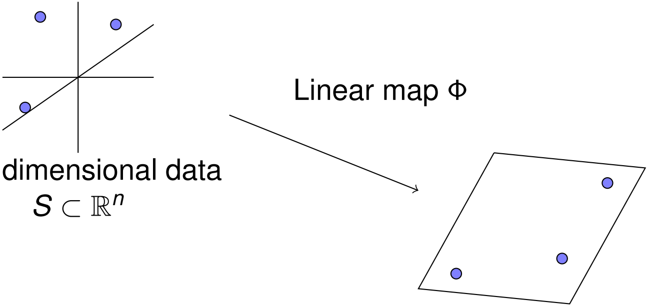

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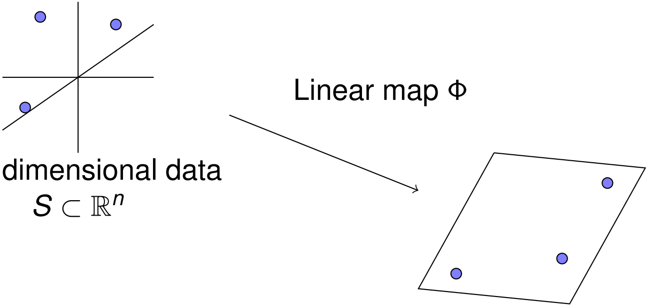

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$$\langle \Phi x, \Phi y\rangle = \langle x, y\rangle \pm \epsilon \|x\|_2 \|y\|_2$$

Johnson-Lindenstrauss Lemma

Theorem (variant of Johnson-Lindenstrauss '84)

Let $x \in \mathbb{R}^n$. A random Gaussian matrix Φ will have

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Set $\delta = 1/2^k$: embed 2^k points into $O(k)$ dimensions.

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- Fast multiplication.
  - Approximate numerical algebra problems (e.g., linear regression, low-rank approximation)
  - k-means clustering

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- And by [BDDW '08], JL ⇒ RIP; so *equivalent*.[1]

---

[1] Round trip loses $\log n$ factor in dimension

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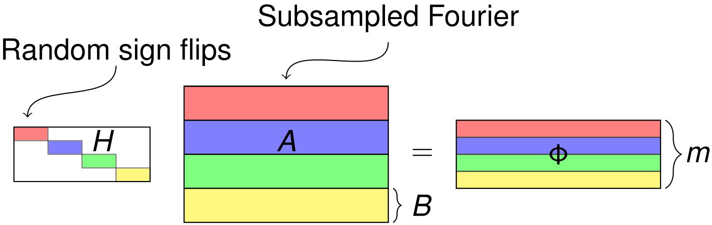
Our result: a fast RIP matrix with fewer rows



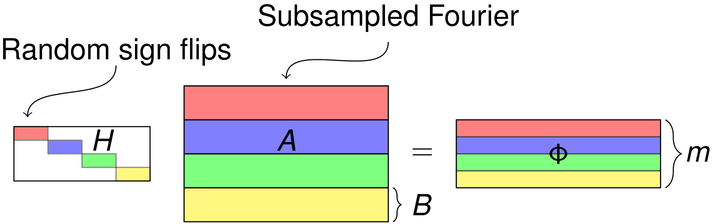
- New construction of fast RIP matrices: sparse times Fourier.
- $k \log^3 n$ rows and $n \log n$ multiplication time.

Theorem

If $m \simeq k \log^3 n$, $B \simeq \log^c n$, and A is a random partial Fourier matrix, then Φ has the RIP with probability at least $2/3$.

Generalization

Our approach is actually works for more general A:



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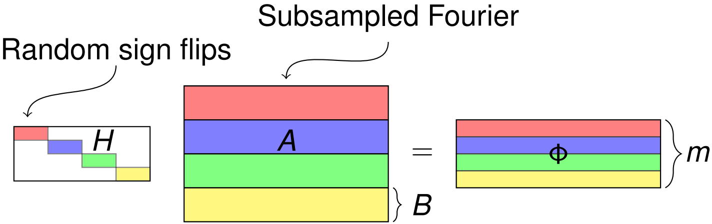

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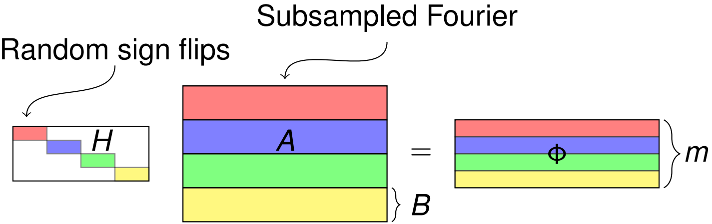

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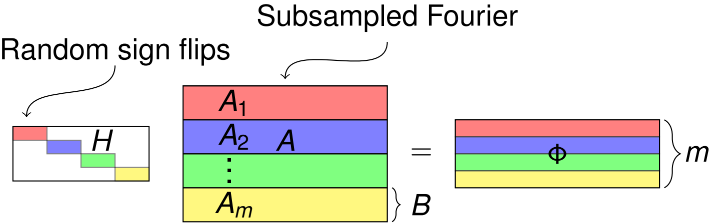

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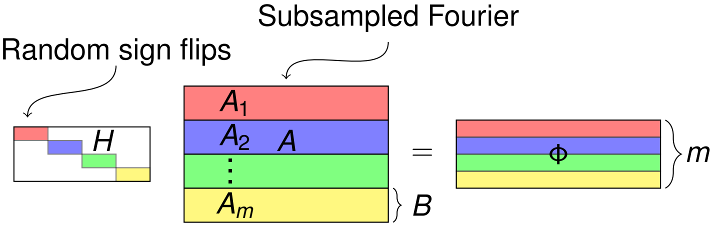

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Then Φ is a good RIP matrix:

- Φ has the RIP (whp) with $m = O(k \log^3 n)$ rows.
- Time to multiply by Φ = time to multiply by $A + mB$.

Results in the area

Construction	Measurements m	Multiplication Time
Sparse JL matrices [KN12]	$\frac{1}{\epsilon^2} k \log n$	ϵmn
Partial Fourier [RV08,CGV13]	$\frac{1}{\epsilon^2} k \log^4 n$	$n \log n$
Partial Circulant [KMR12]	$\frac{1}{\epsilon^2} k \log^4 n$	$n \log n$
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Dimension:	$n \xrightarrow{\hspace{3em}} k \log^4 n$
Time:	$n \log n$

[RV08]

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Dimension:	$n \longrightarrow k \log^4 n$	$\longrightarrow k \log n$
Time:	$n \log n$	$k^2 \log^5 n$
	[RV08]	Gaussian

Concentration of Measure

Let Σ_k is unit-norm k-sparse vectors.
We want to show for our distribution Φ on matrices that

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Probabilists have lots of tools to analyze this.

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Tools

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Screwdriver

Tools

Screwdriver

Drill

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Screwdriver

Bit sets

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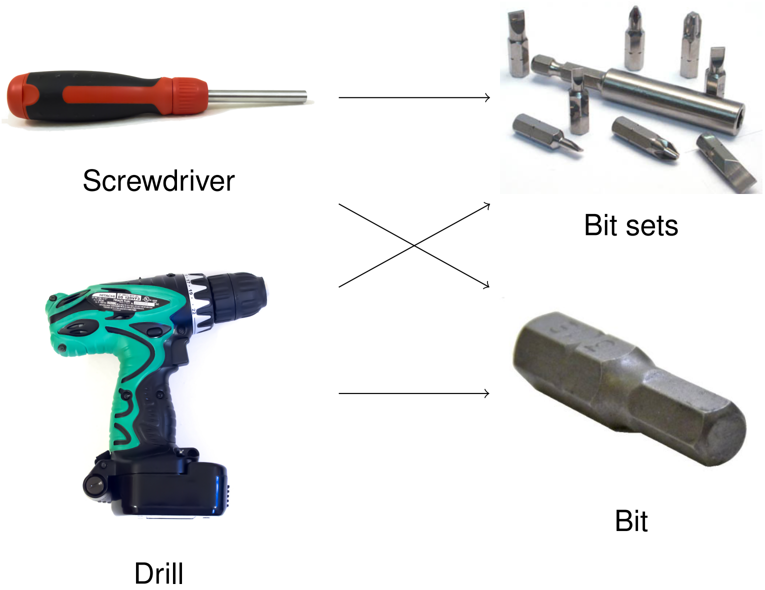

Tools

Common interface: m drivers, n bits $\implies$ mn combinations.

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Hex shanks

Common interface
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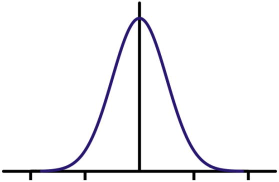


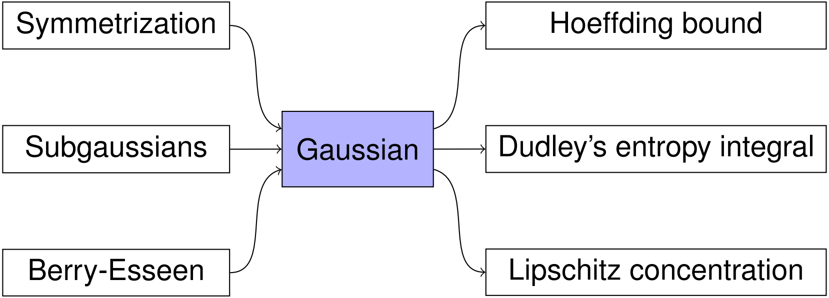
Gaussians

Common interface
for probability

A Probabilist's Toolbox

Convert to Gaussians

Gaussian concentration



Symmetrization

Subgaussians

Berry-Esseen

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Hoeffding bound

Dudley's entropy integral

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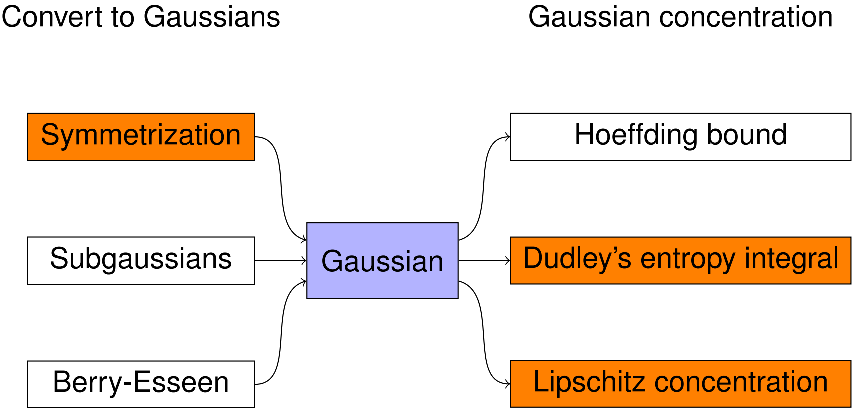

Convert to Gaussians

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Will prove: symmetrization and Dudley's entropy integral.

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Symmetrization

Lemma (Symmetrization)

Suppose $X_1, \ldots, X_t$ are i.i.d. with mean μ. For any norm $\|\cdot\|$,

$$\mathbb{E}\left[\left\|\frac{1}{t}\sum_i X_i - \mu\right\|\right] \leq 2\,\mathbb{E}\left[\left\|\frac{1}{t}\sum_i s_i X_i\right\|\right]$$

where $s_i \in \{\pm 1\}$ independently.

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and apply the triangle inequality. □

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where $s_i \in \{\pm 1\}$ independently and $g_i \sim N(0, 1)$ independently.

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We have $\mathbb{E}[|g_i|] \approx .8 > 2/3$.

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Gaussian Processes

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Let A be a random $m \times n$ Gaussian matrix. For any $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, define

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- In example: $\|(u, v) - (u', v')\| = \|uv^T - u'v'^T\|_F$.

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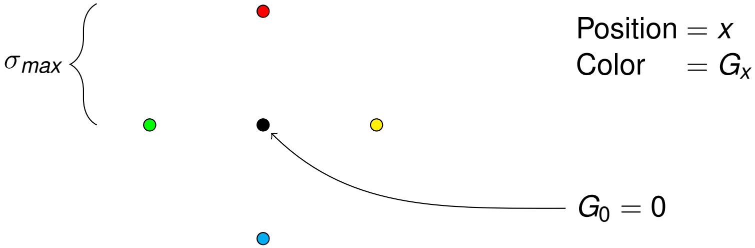

σ_{max}

Position $= x$
Color $= G_x$

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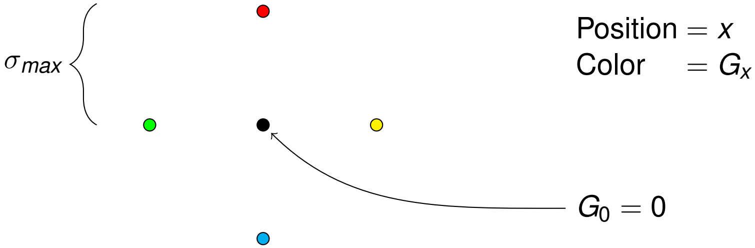

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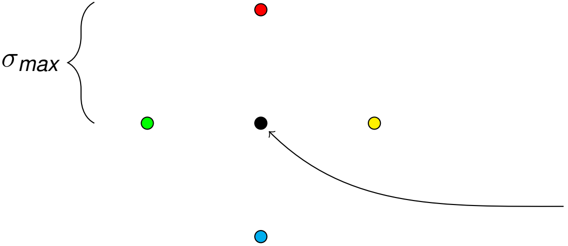

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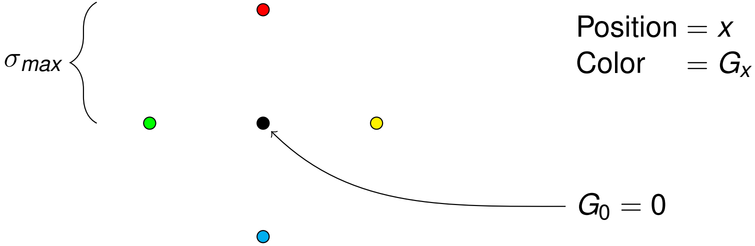

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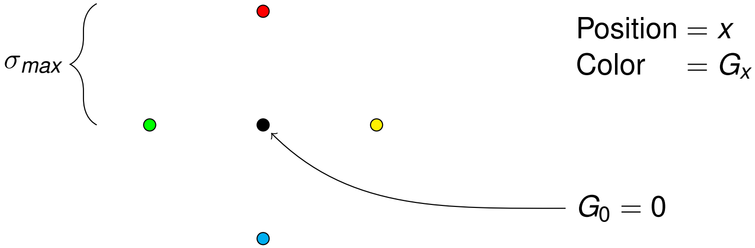

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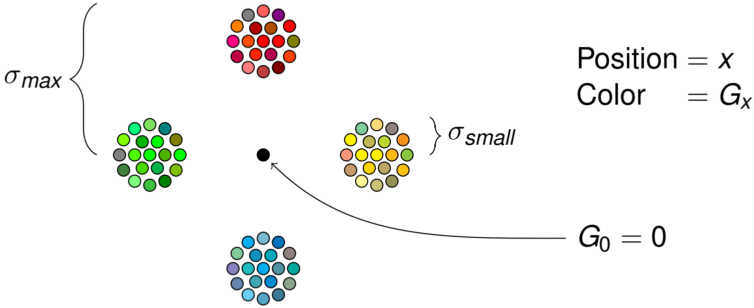

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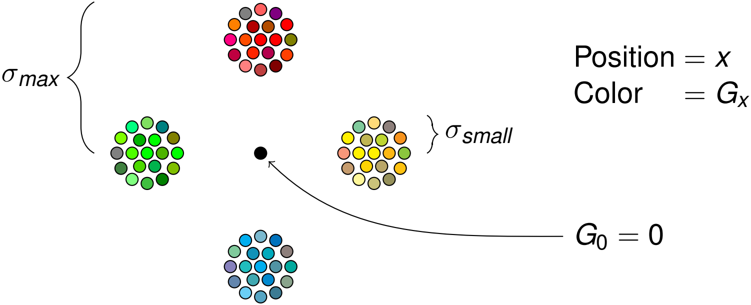
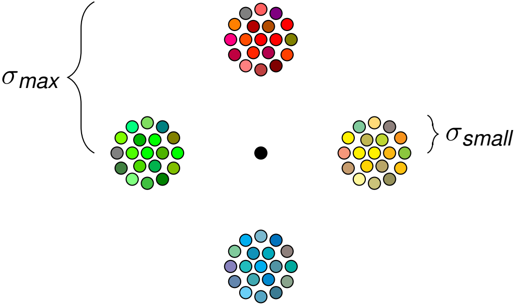
  - $\mathbb{E} \sup_{x \in T} G_x \lesssim \sigma_{max} \sqrt{\log n}$
- Two levels: $\sigma_{max} \sqrt{\log 4} + \sigma_{small} \sqrt{\log n}$.



Gaussian Processes: chaining

- Bound $\mathbb{E}\sup_{x \in T} G_x$, where $G_x - G_y$ has variance $\|x - y\|^2$.
- Two levels: $\sigma_{max}\sqrt{\log 4} + \sigma_{small}\sqrt{\log n}$.



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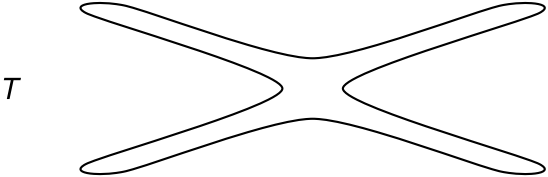

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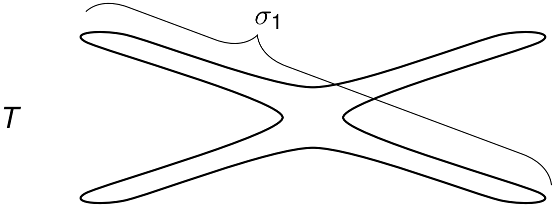

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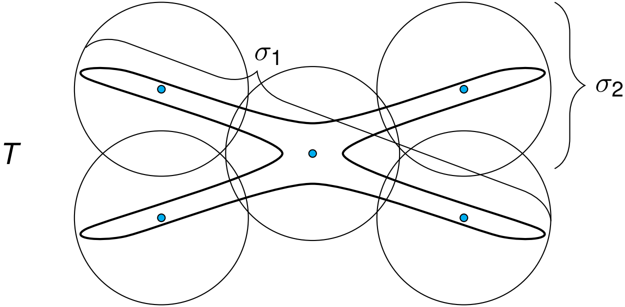

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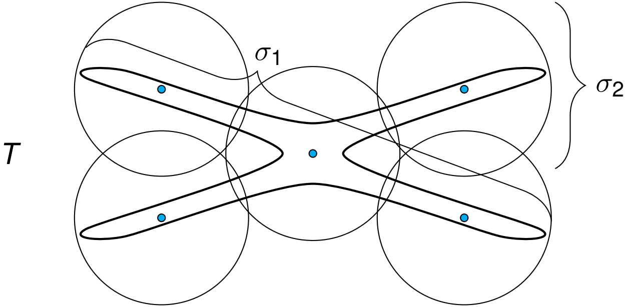

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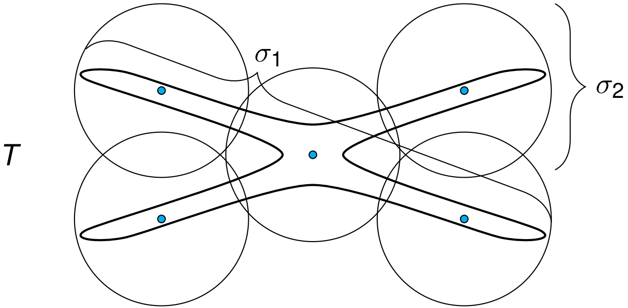

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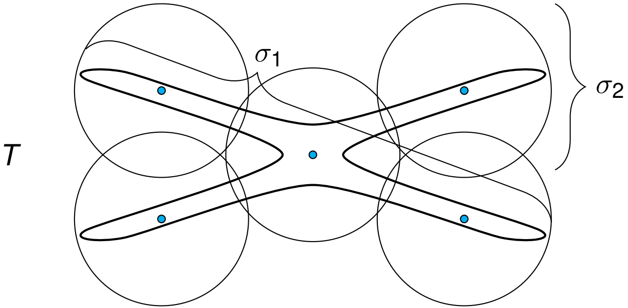

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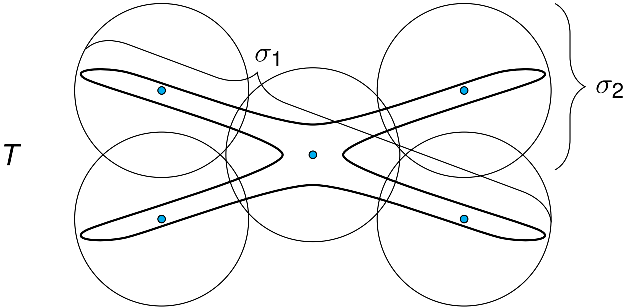

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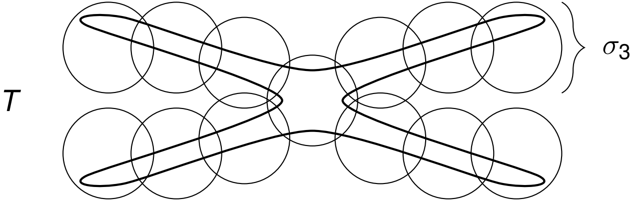

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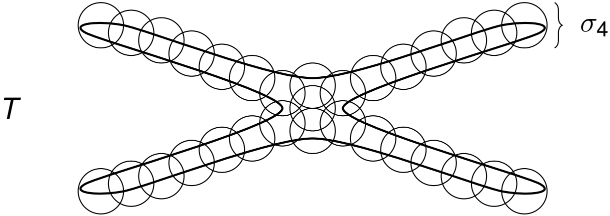

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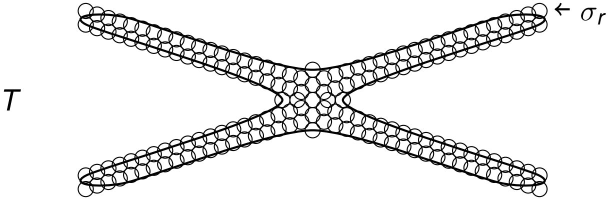

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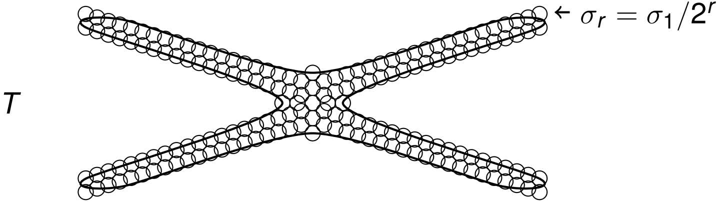

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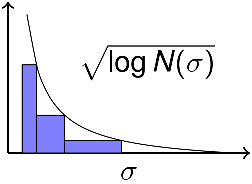

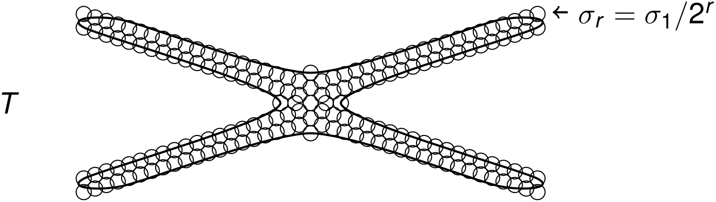

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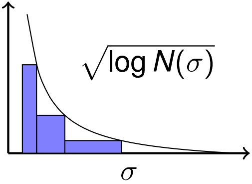

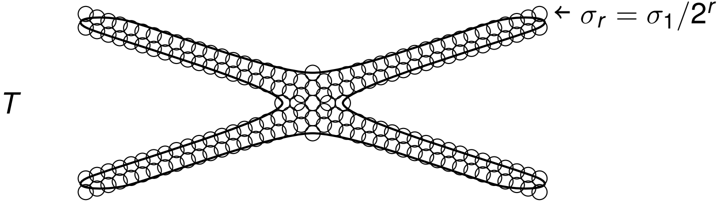

Gaussian Processes

Dudley's Entropy Integral, Talagrand's generic chaining

Theorem (Dudley's Entropy Integral)

Define the norm $\|\cdot\|$ *of a Gaussian process G by*

$$\|x - y\| = \textit{standard deviation of } (G_x - G_y).$$

Then

$$\mathbb{E} \sup_{x \in T} G_x \lesssim \int_0^\infty \sqrt{\log N(T, \|\cdot\|, u)} du$$

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- Bound a random variable using geometry.

Outline

1. Introduction
   - Compressive sensing
   - Johnson Lindenstrauss Transforms
   - Our result
2. Concentration of measure: a toolbox
   - Overview
   - Symmetrization
   - Gaussian Processes
   - Lipschitz Concentration
3. Proof
   - Overview
   - Covering Number
4. Conclusion

Lipschitz Concentration of Gaussians

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is C-Lipschitz and $g \sim N(0, I_n)$, then for any $t > 0$,

$$\Pr[f(g) > \mathbb{E}[f(g)] + Ct] \le e^{-\Omega(t^2)}.$$

- f concentrates as well as *individual* Gaussians.
- Can replace f with $-f$ to get lower tail bound.

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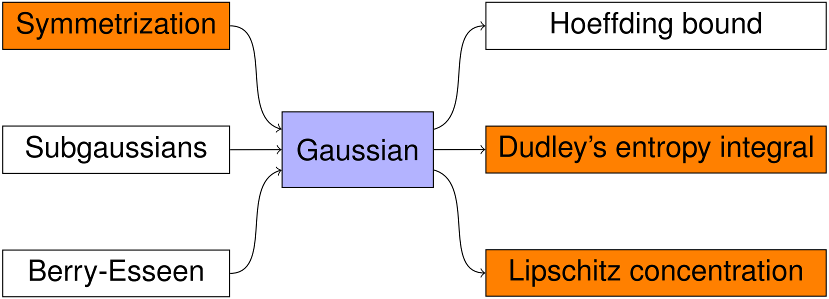
$\implies$ the Johnson-Lindenstrauss lemma.

A Probabilist's Toolbox (recap)

Convert to Gaussians

Gaussian concentration



Symmetrization

Subgaussians

Berry-Esseen

Gaussian

Hoeffding bound

Dudley's entropy integral

Lipschitz concentration

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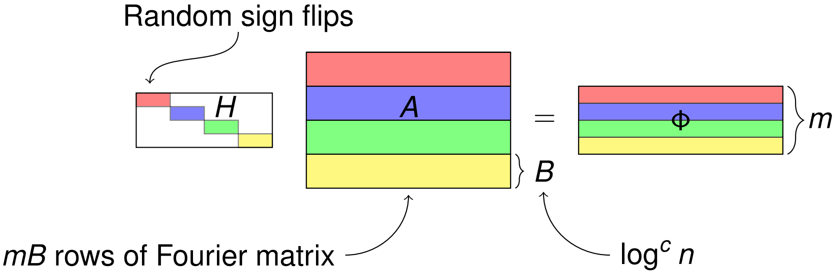
Goal



For Σ_k denoting unit-norm k-sparse vectors, we want

$$\mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|x\|_2^2 \right| < \epsilon,$$

(Expectation of $*$) = $*$

Proof outline: Rudelson-Vershynin

Rudelson-Vershynin: subsampled Fourier, $O(k \log^4 n)$ rows.

$\mathbb{E} \sup$
$\|A^T A - \mathrm{I}\|$

Expected
sup deviation

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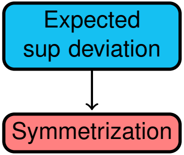

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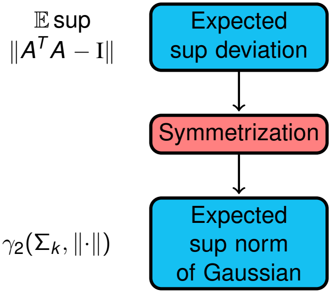

γ_2 : supremum of Gaussian process

Σ_k : k-sparse unit vectors

$\|\cdot\|$: a norm that depends on A (specified in a few slides)

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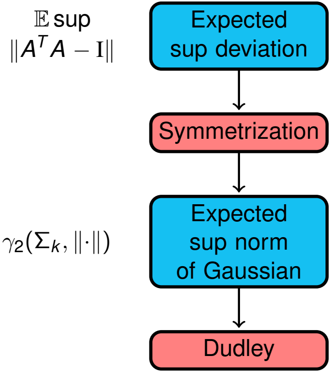

$\mathbb{E} \sup$
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Expected sup deviation

Symmetrization

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

Dudley

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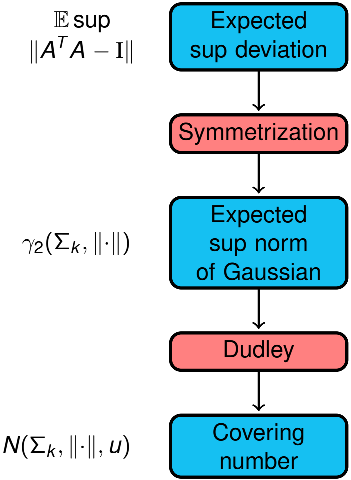

$\mathbb{E} \sup \|A^T A - \mathrm{I}\|$ — Expected sup deviation

Symmetrization

$\gamma_2(\Sigma_k, \|\cdot\|)$ — Expected sup norm of Gaussian

Dudley

$N(\Sigma_k, \|\cdot\|, u)$ — Covering number

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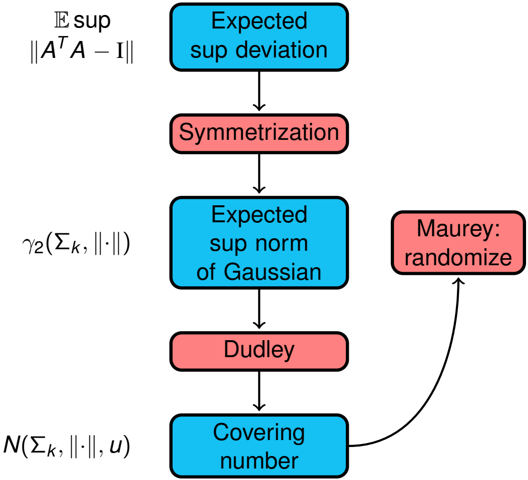

$\mathbb{E} \sup \|A^T A - \mathrm{I}\|$

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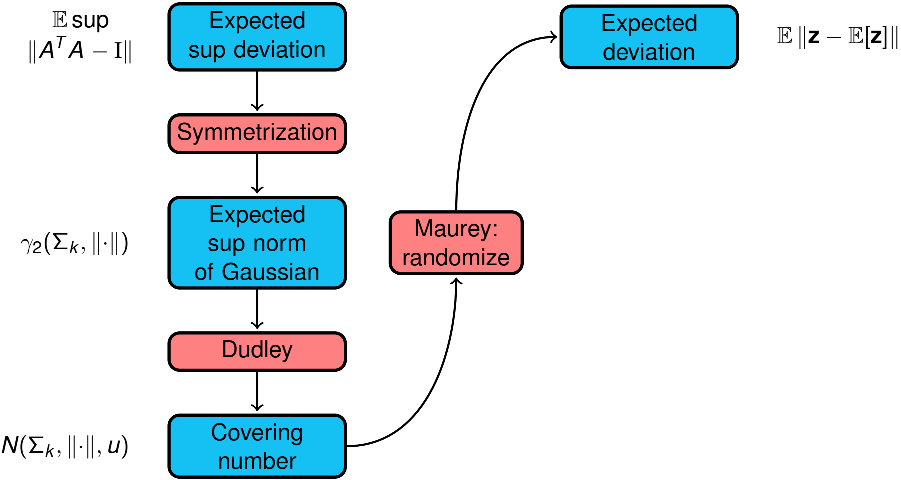

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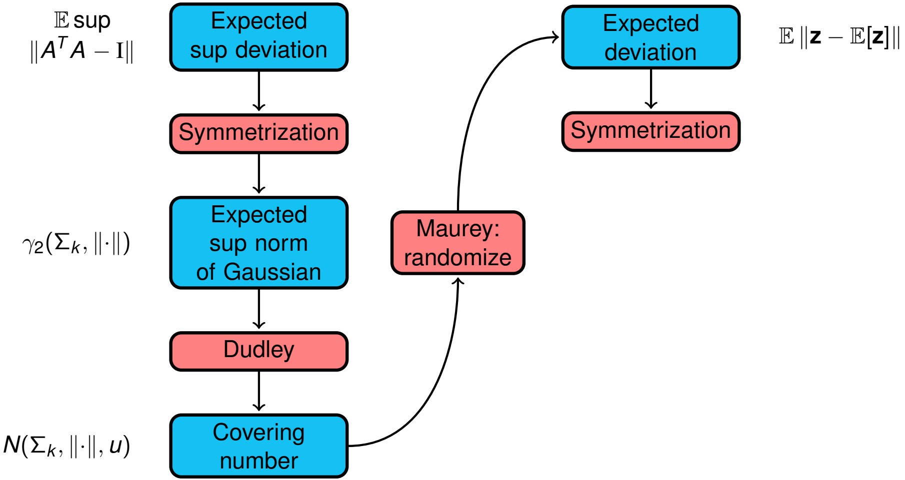

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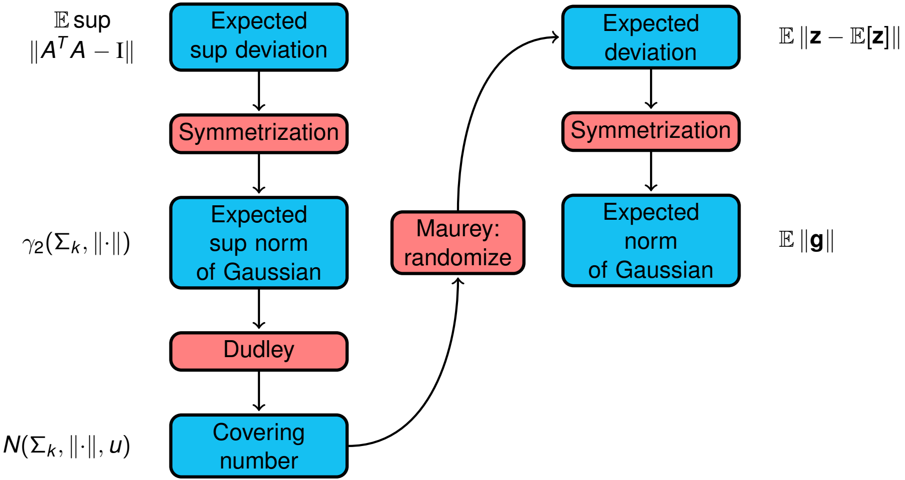

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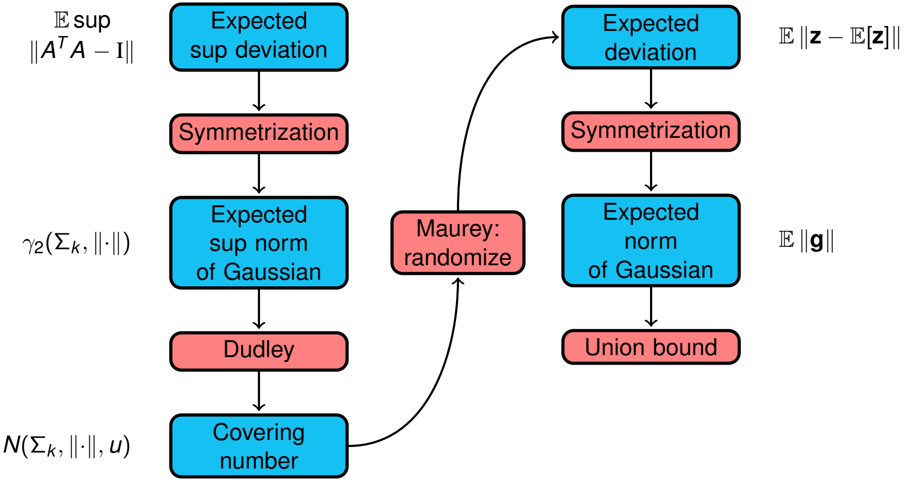

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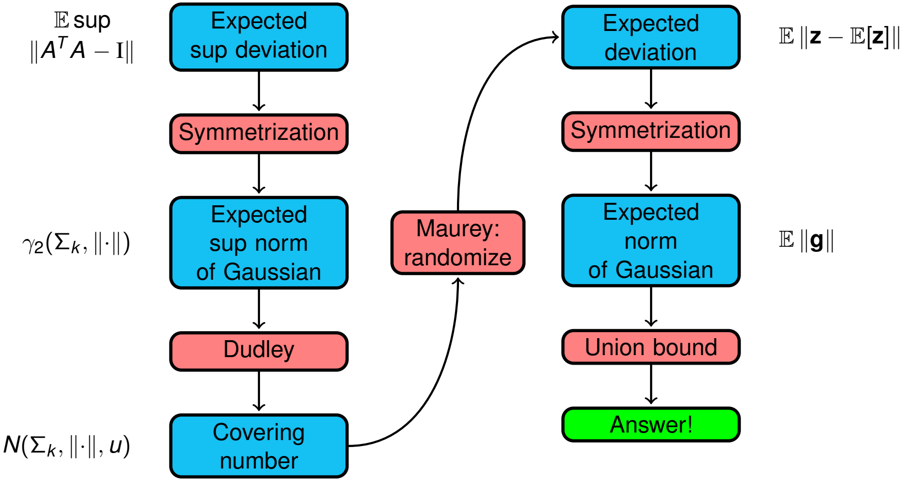

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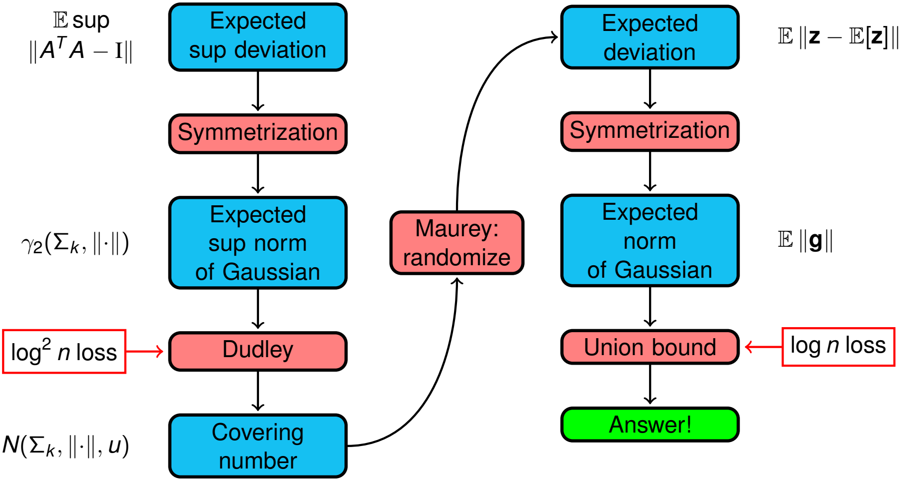


$\mathbb{E}\sup \|A^T A - \mathrm{I}\|$

Expected sup deviation

Symmetrization

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

log^2 *n* loss

Dudley

$N(\Sigma_k, \|\cdot\|, u)$

Covering number

Maurey: randomize

Expected deviation

$\mathbb{E}\|\mathbf{z} - \mathbb{E}[\mathbf{z}]\|$

Symmetrization

Expected norm of Gaussian

$\mathbb{E}\|\mathbf{g}\|$

Union bound

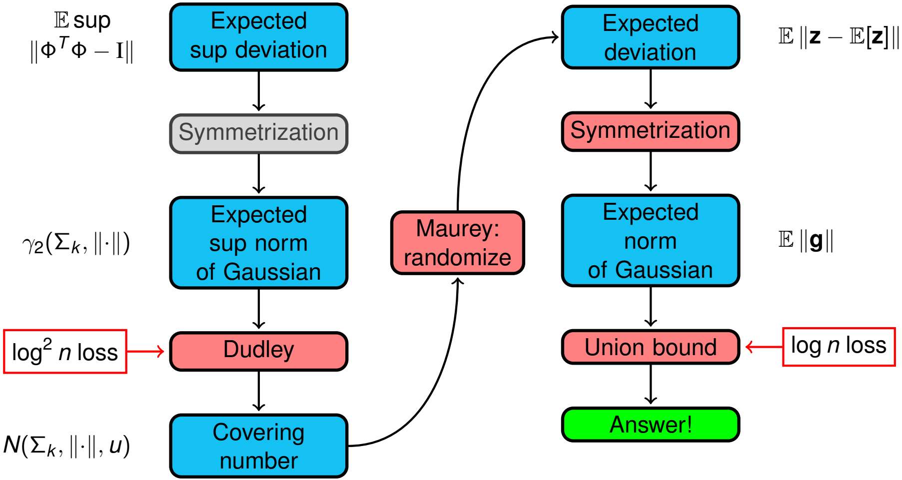
log *n* loss

Answer!

Proof outline

Rudelson-Vershynin: subsampled Fourier, $O(k \log^4 n)$ rows.

Nelson-P-Wootters: sparse times Fourier, $O(k \log^3 n)$ rows.



$\mathbb{E} \sup \|\Phi^T\Phi - I\|$

Expected sup deviation

Symmetrization

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

$\log^2 n$ loss

Dudley

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Covering number

Maurey: randomize

$\mathbb{E}\|\mathbf{z} - \mathbb{E}[\mathbf{z}]\|$

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$\mathbb{E}\|\mathbf{g}\|$

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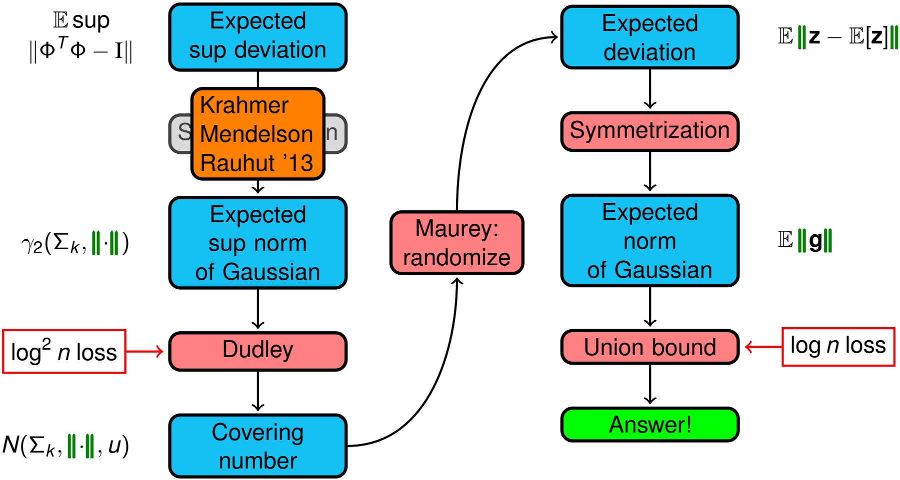

$\mathbb{E} \sup \|\Phi^T \Phi - I\|$

Expected sup deviation

S Krahmer Mendelson Rauhut '13 n

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

log^2 n loss

Dudley

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Covering number

Maurey: randomize

$\mathbb{E}\|\mathbf{z} - \mathbb{E}[\mathbf{z}]\|$

Expected deviation

Symmetrization

$\mathbb{E}\|\mathbf{g}\|$

Expected norm of Gaussian

log n loss

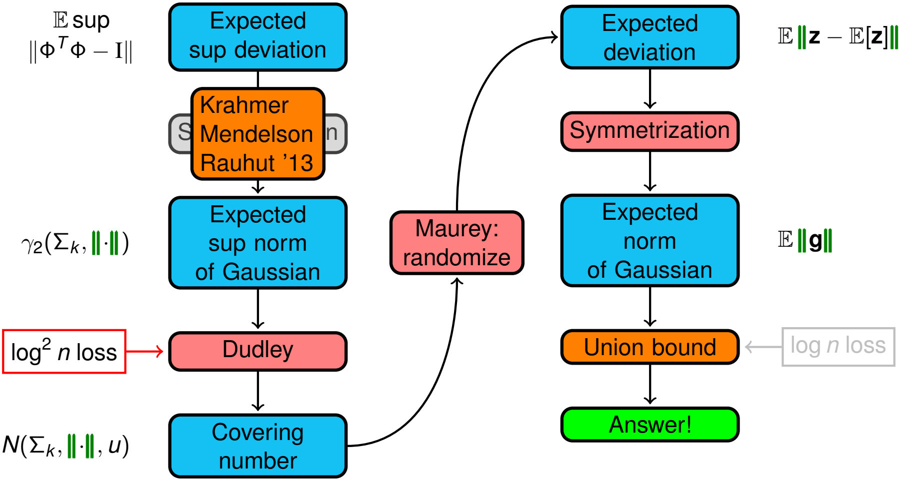
Union bound

Answer!

Proof outline

Rudelson-Vershynin: subsampled Fourier, $O(k \log^4 n)$ rows.

Nelson-P-Wootters: sparse times Fourier, $O(k \log^3 n)$ rows.



$\mathbb{E} \sup \|\Phi^T \Phi - I\|$

Expected sup deviation

S n

Krahmer Mendelson Rauhut '13

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

log² n loss

Dudley

$N(\Sigma_k, \|\cdot\|, u)$

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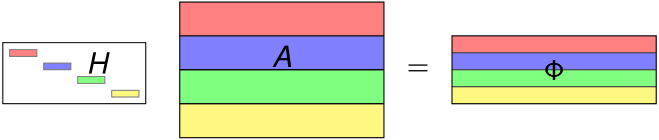
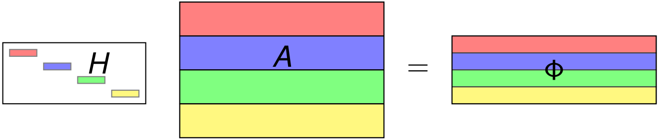
Proof part I: triangle inequality



$$\mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|x\|_2^2 \right|$$

$$\leq \mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|Ax\|_2^2 \right| + \mathbb{E} \sup_{x \in \Sigma_k} \left| \|Ax\|_2^2 - \|x\|_2^2 \right|$$

Proof part I: triangle inequality



$$
\begin{aligned}
&\mathbb{E}\sup_{x\in\Sigma_k}\left|\|\Phi x\|_2^2-\|x\|_2^2\right| \\
&\quad\le \mathbb{E}\sup_{x\in\Sigma_k}\left|\|\Phi x\|_2^2-\|Ax\|_2^2\right|+\mathbb{E}\sup_{x\in\Sigma_k}\left|\|Ax\|_2^2-\|x\|_2^2\right| \\
&\quad= \mathbb{E}\sup_{x\in\Sigma_k}\left|\|X_As\|_2^2-\mathbb{E}_s\|X_As\|_2^2\right|+(\text{RIP constant of }A),
\end{aligned}
$$

where X_A is some matrix depending x and A, and s is the vector of random sign flips used in H.

Proof part I: triangle inequality

$$\mathbb{E}\sup_{x\in\Sigma_k}\left|\|X_A s\|_2^2 - \mathbb{E}_s\|X_A s\|_2^2\right| + (\text{RIP constant of } A)$$

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By assumption, this is small.
(Recall *A* has extra rows)

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This is a *Rademacher Chaos Process.*
We have to do some work to show that it is small.

Proof part II: probability and geometry

By [KMR12] and some manipulation, can bound the Rademacher chaos using

$$\gamma_2(\Sigma_k, \| \cdot \|_A)$$

Some norm induced by A

The supremum of a Gaussian process over Σ_k with norm $\|\cdot\|_A$

Dudley's entropy integral: can estimate this by bounding the *covering number* $N(\Sigma_k, \|\cdot\|_A, u)$.

Definition of the Norm

$$N(\Sigma_k, \|\cdot\|_A, u)$$

for the norm $\|x\|_A$:

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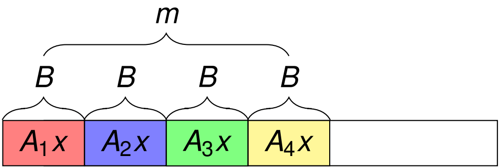
Ax
$= \text{subset of } \widehat{x}$



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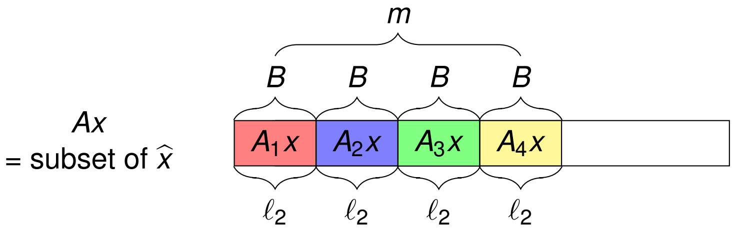

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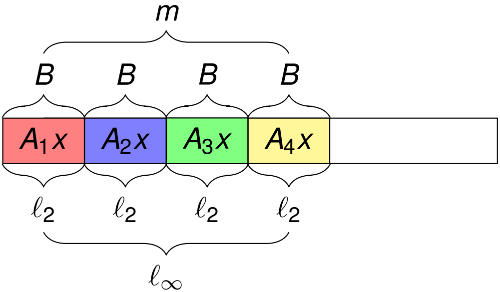

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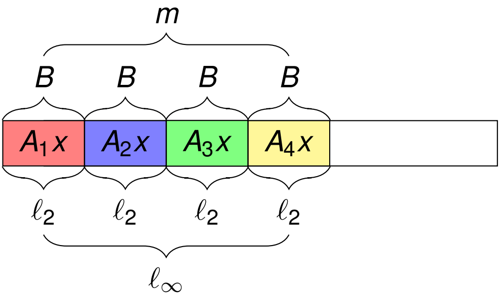

$$\|x\|_A = \max_{i \in [m]} \|A_i x\|_2.$$

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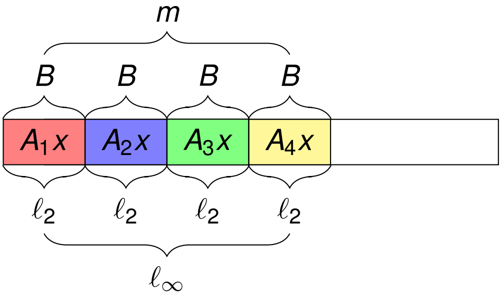

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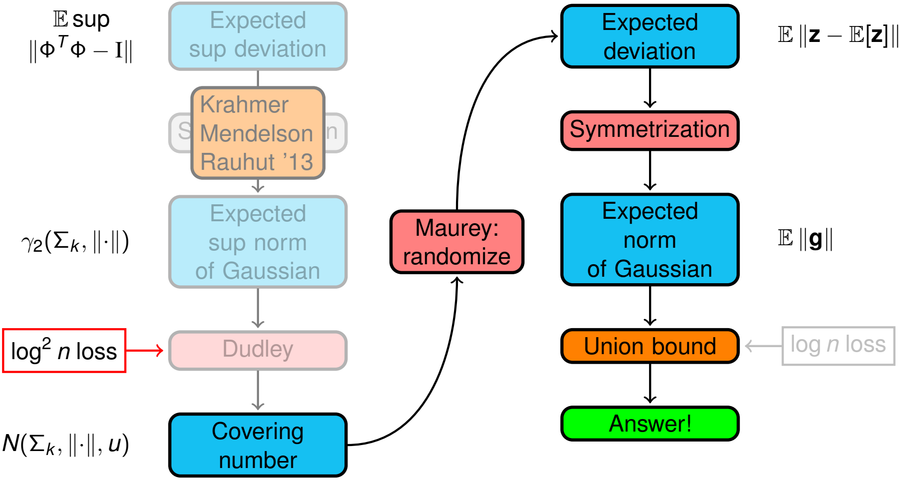
Rudelson-Vershynin: estimates $N(\Sigma_k, \|\cdot\|_A, u)$ when $B = 1$.

Progress



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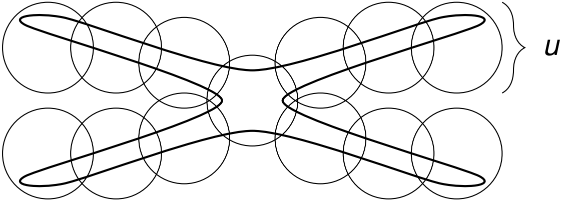
Answer!

Outline

1. Introduction
   - Compressive sensing
   - Johnson Lindenstrauss Transforms
   - Our result
2. Concentration of measure: a toolbox
   - Overview
   - Symmetrization
   - Gaussian Processes
   - Lipschitz Concentration
3. Proof
   - Overview
   - Covering Number
4. Conclusion

Covering Number Bound

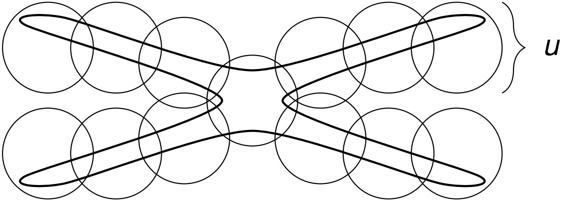
$$N(\Sigma_k, \|\cdot\|_A, u)$$



$$\Sigma_k \quad = \{k\text{-sparse } x \mid \|x\|_2 \leq 1\}$$

Covering Number Bound

$$N(\Sigma_k, \|\cdot\|_A, u) \leq N(B_1, \|\cdot\|_A, u/\sqrt{k})$$



$$\begin{aligned} \Sigma_k &= \{k\text{-sparse } x \mid \|x\|_2 \leq 1\} \\ \subset \sqrt{k}B_1 &= \{x \mid \|x\|_1 \leq \sqrt{k}\} \end{aligned}$$

Covering number bound

$$N(B_1, \|\cdot\|_A, u)$$

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Covering Number Bound

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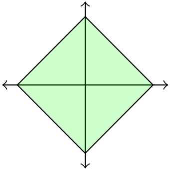

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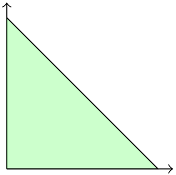
Covering Number Bound

Maurey's empirical method



- How many balls of radius u required to cover B_1^+?

Covering Number Bound

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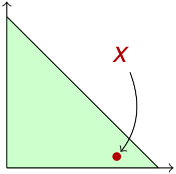
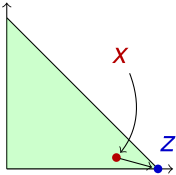


- How many balls of radius u required to cover B_1^+?
- Consider any $x \in B_1^+$.

Covering Number Bound

Maurey's empirical method



- How many balls of radius u required to cover B_1^+?
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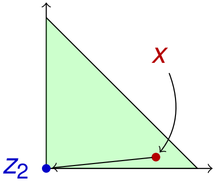

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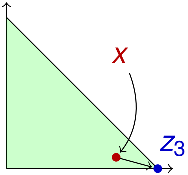

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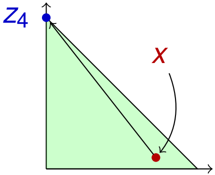

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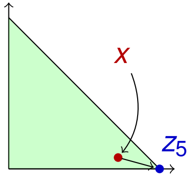

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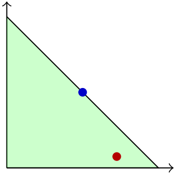

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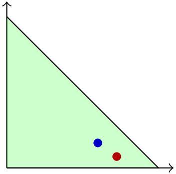

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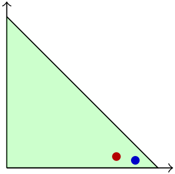

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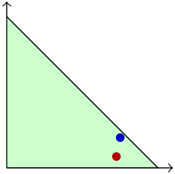

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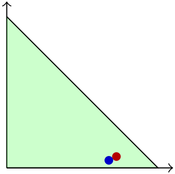

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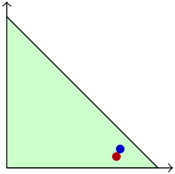

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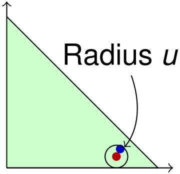

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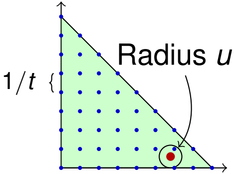
- Let t be large enough that, regardless of x,

$$\mathbb{E}[\|\mathbf{z} - x\|] \leq u.$$

Covering Number Bound

Maurey's empirical method



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Covering Number Bound

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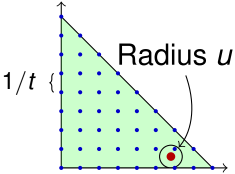

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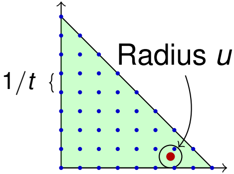
  - Then $N(B_1, \|\cdot\|, u) \leq$ number of $\mathbf{z}$

Covering Number Bound

Maurey's empirical method



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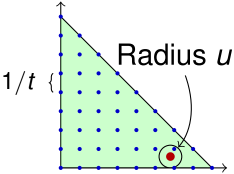
- All x lie within u of at least one possible $\mathbf{z}$.
  - Then $N(B_1, \|\cdot\|, u) \leq$ number of $\mathbf{z} \leq (n+1)^t$.
  - Only $(n+1)^t$ possible tuples $(z_1, \ldots, z_t) \implies \mathbf{z}$.

Covering Number Bound

Maurey's empirical method



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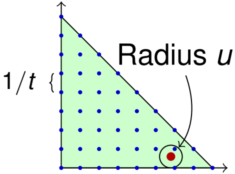

Will show: $\mathbb{E}[\|\mathbf{z} - x\|_A] \leq \sqrt{B/t}$

- Let $z_1, \ldots, z_t$ be i.i.d. randomized roundings of x to simplex.
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$$\mathbb{E}[\|\mathbf{z} - \mathbf{x}\|] \leq \mathbf{u}.$$

- All x lie within u of at least one possible $\mathbf{z}$.
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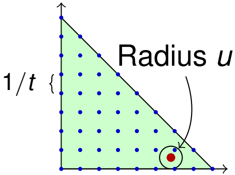
  - Only $(n+1)^t$ possible tuples $(z_1, \ldots, z_t) \implies \mathbf{z}$.

Covering Number Bound

Maurey's empirical method



Will show: $\mathbb{E}[\|\mathbf{z} - x\|_A] \leq \sqrt{B/t} \implies N(T, \|\cdot\|_A, u) \leq n^{B/u^2}$

- Let $z_1, \ldots, z_t$ be i.i.d. randomized roundings of x to simplex.
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$$\begin{aligned}\mathbb{E}[\|\frac{1}{t}\sum z_i - x\|_A] &\lesssim \mathbb{E}[\|\frac{1}{t}\sum g_i z_i\|_A] \\ &=: \frac{1}{\sqrt{t}}\mathbb{E}[\|\mathbf{g}\|_A]\end{aligned}$$

where $\mathbf{g} \in \mathbb{R}^n$ has

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independently in each coordinate.

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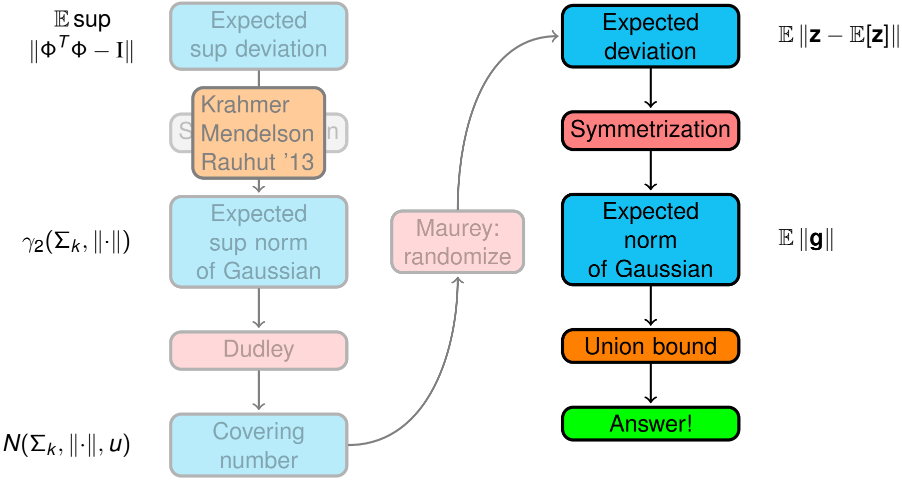
- (Note: $\mathbb{E}[\|\mathbf{g}\|_2] \leq 1 \implies N(B_1, \ell_2, u) \leq n^{1/u^2}$.)

Progress


$\mathbb{E}\sup$
$\|\Phi^T\Phi - \mathrm{I}\|$
Expected sup deviation
S
n
Krahmer Mendelson Rauhut '13
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Expected sup norm of Gaussian
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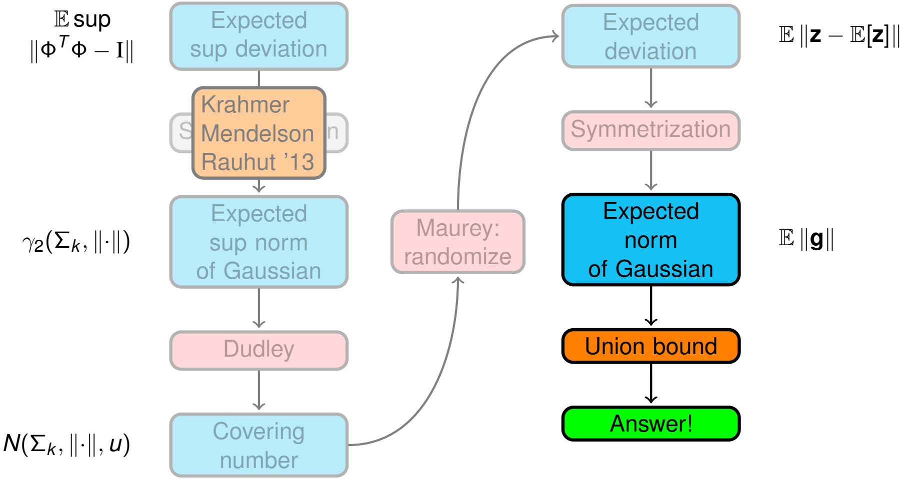
Answer!

Progress



$\mathbb{E}$ sup $\|\Phi^T\Phi - I\|$

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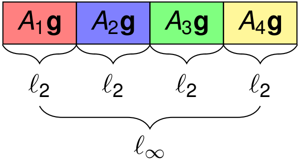

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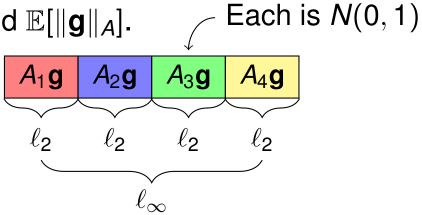

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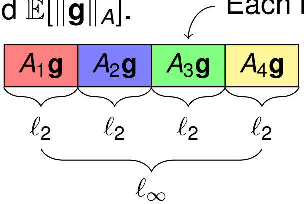
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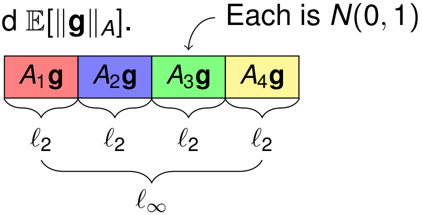

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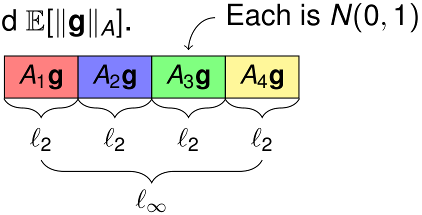

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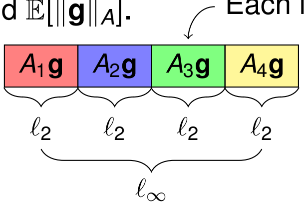

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$\uparrow$

Lipschitz concentration
(just like $\sqrt{n} + \sqrt{\log(1/\delta)}$ in tutorial)

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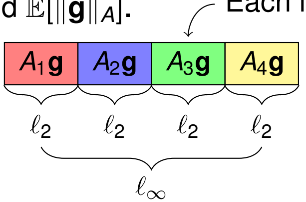

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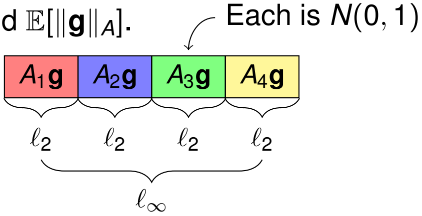

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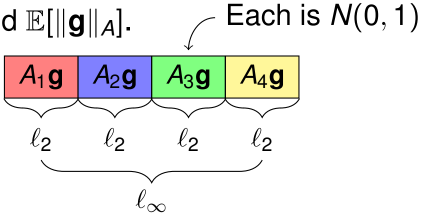

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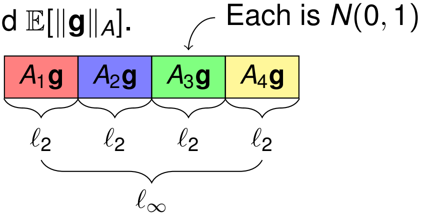

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Bounding the norm (by example)

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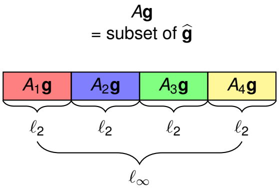

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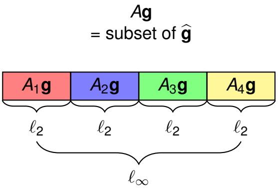

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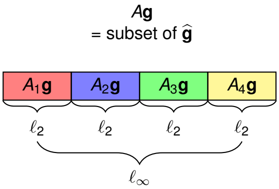

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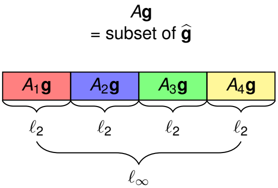

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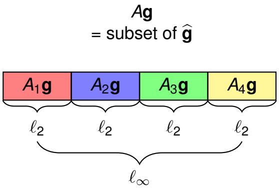

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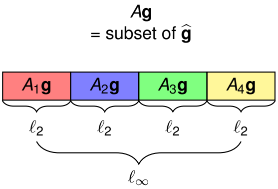

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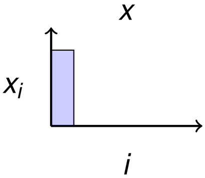

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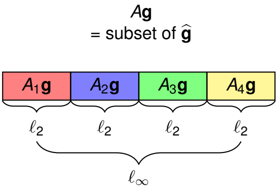

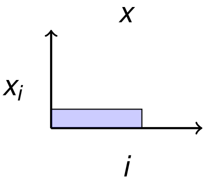

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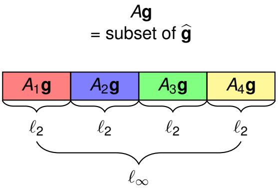

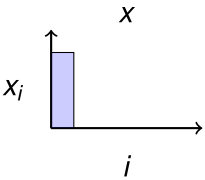

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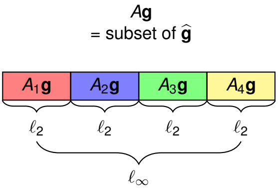
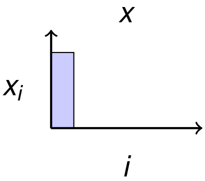
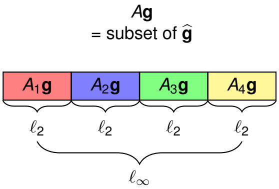




Example (1):

- $\|x\|_1 \ll 1$.

Bounding the norm (by example)



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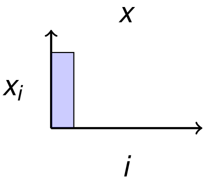

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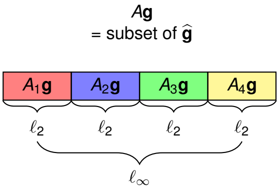

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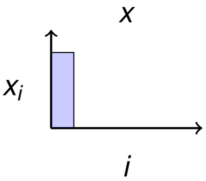

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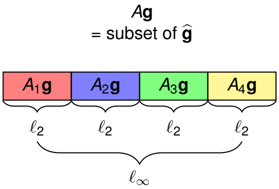

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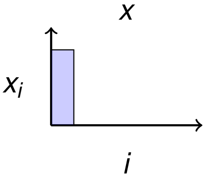

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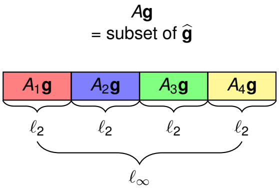

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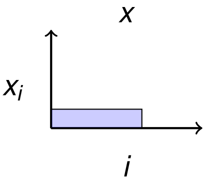

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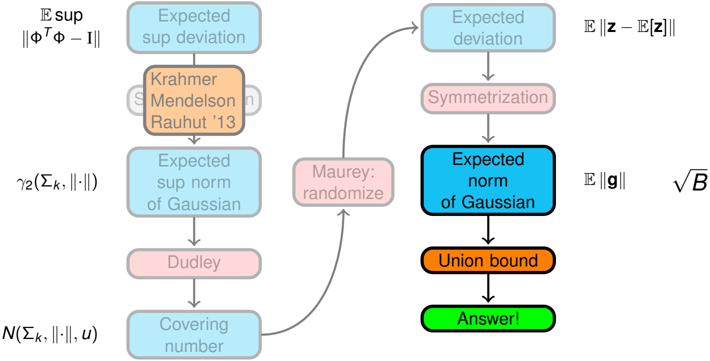

Union bound just loses a constant factor

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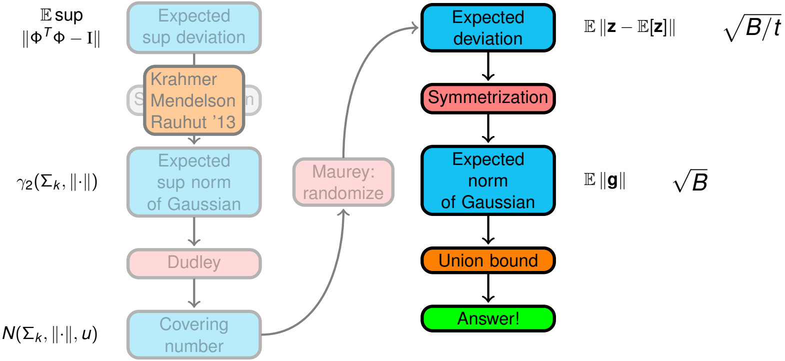

$\mathbb{E}\sup \|\Phi^T\Phi - I\|$

Expected sup deviation

Krahmer Mendelson Rauhut '13

$\gamma_2(\Sigma_k, \|\cdot\|)$

Expected sup norm of Gaussian

Dudley

$N(\Sigma_k, \|\cdot\|, u)$

Covering number

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Symmetrization

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Answer!

Sample mean $\mathbf{z}$ expects to lie within u of $\mathbf{x}$ for $t \geq B/u^2$

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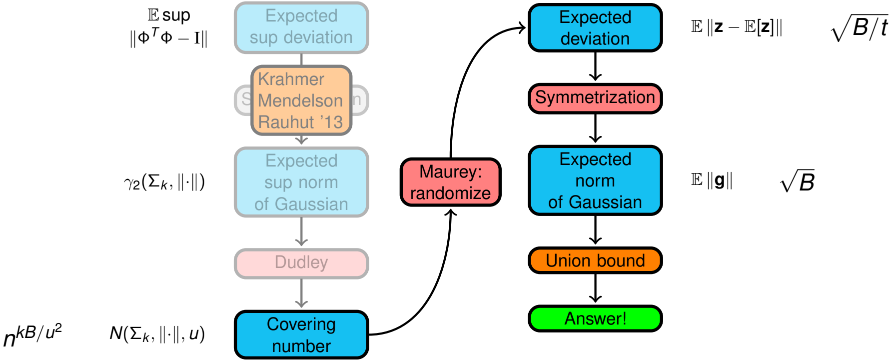

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Covering number of B_1 is $(n+1)^{B/u^2}$

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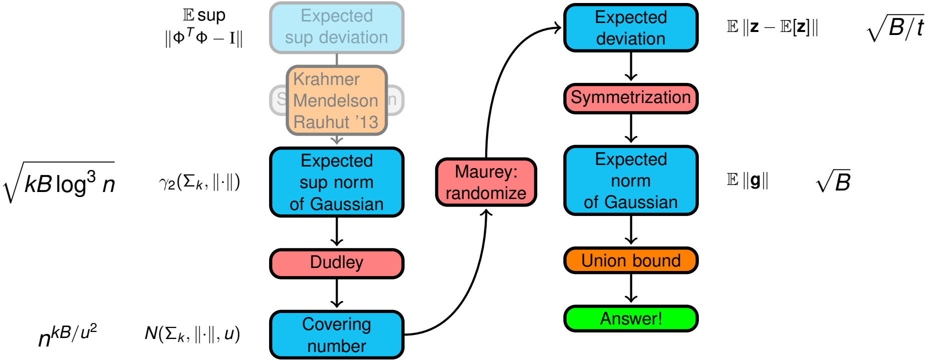

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Entropy integral is $\sqrt{kB\log^3 n}$

Unrolling everything

$\sqrt{\frac{k \log^3 n}{m}}$ $\mathbb{E}\sup \|\Phi^T\Phi - \mathrm{I}\|$

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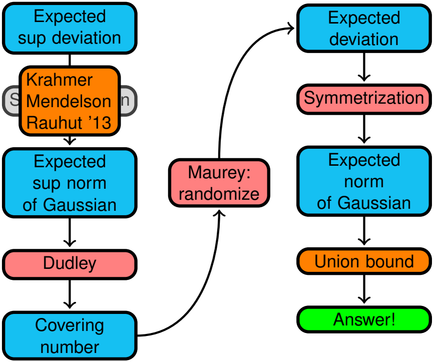

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  giving an $n^{O(1/u^2)}$ bound.

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- $x \in \Sigma_k/\sqrt{k} \subset B_1$ rounded to $z_1, \ldots, z_t$ symmetrized to g.

$$\mathcal{G}(x) = \mathbb{E}_{z,g}\|g\|_A$$

- First: split x into “large” and “small” coordinates.

$$\mathcal{G}(x) \le \mathcal{G}(x_{large}) + \mathcal{G}(x_{small})$$

- x_{large}: Locations where $x_i > (\log n)/k$
  - Bound:

$$\|x_{large}\|_1 \le 1/\log n.$$

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- So can focus on $\|x\|_\infty < (\log n)/k$.

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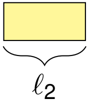

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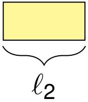

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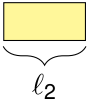

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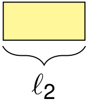

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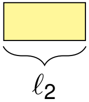

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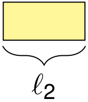

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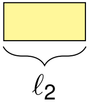

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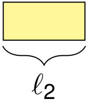

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- So with high probability, $\|A_i g\|_2 \lesssim \sqrt{B/t} + C\sqrt{\log n} \lesssim \sqrt{B/t}$.
- So $\mathbb{E}\|g\|_A = \max \|A_i g\|_2 \lesssim \sqrt{B/t}$.