Programming curvature using origami tessellations

Levi Dudte, Etienne Vouga, Tomohiro Tachi, L. Mahadevan

Contents

1 Geometry of the Miura-ori 2

2 Constructing Generalized Miura-ori tessellations 3
   2.1 Explicit Construction for Generalized Cylinders 3
   2.2 Curved Surfaces 5
      2.2.1 Initial Positions 7
      2.2.2 Fixed Nodes 7
      2.2.3 Developability Constraints 8
      2.2.4 Flat-foldability Constraints 8
      2.2.5 Special Cases 9
      2.2.6 Objective Function 10
      2.2.7 Numerical Optimization Approach 10

3 Examples 11

4 Foldability 11
   4.1 Simulation Method 13
   4.2 Structural Mechanics of Origami 15
   4.3 Experimental measurement of Hyper stiffness 16

5 Accuracy vs. Effort: Hyperboloid of a Single Sheet 17
   5.1 Base Mesh: Diagonal, Rotationally Symmetric Strip 18
   5.2 Examples 19
   5.3 Computing the Hausdorff Distance 20
   5.4 Accuracy/Effort Trade-Off 22
   5.5 Area Convergence 22
1 Geometry of the Miura-ori

An origami tessellation made of unit cells composed of four quadrilaterals, as in the Miura-ori pattern, but whose unit cells are not necessarily congruent but vary in shape across the tessellation, will be termed a *generalized Miura-ori pattern*. An embedding of such a pattern in \( \mathbb{R}^3 \) can be represented as a quadrilateral mesh: a set of vertices \( p_i \in \mathbb{R}^3 \), edges connecting the vertices and representing the Miura-ori creases, and faces, with four faces meeting at each interior vertex. Given an arbitrary quadrilateral mesh of regular valence four, when is it an isometric embedding of some generalized Miura-ori tessellation? Two constraints are evident: each quadrilateral face must be planar, and the neighborhood of each vertex must be *developable*, i.e. the interior angles around that vertex must sum to \( 2\pi \). It is also easy to see that these conditions are sufficient, in the case that the tessellation is assumed to be topologically trivial.

A quadrilateral mesh that satisfies these conditions is an isometrically embedded generalized Miura-ori tessellation, but what about the four additional properties listed in the introduction?

- **One degree of freedom**: a unit cell with four planar quads in generic position, i.e. whose four creases have nonzero turning angle, has only one degree of freedom. This local property bounds the possible global isometric deformations: the Miura-ori pattern, if it is rigid-foldable at all, has only one degree one freedom, except in the degenerate case where one of more of its hinges have zero turning angle.

- **Negative Poisson’s ratio**: this property again can be understood by examining a single unit cell: a rigid-foldable unit cell must consist of three valley and one mountain crease, or vice-versa, and hence folds with negative Poisson’s ratio.

- **Rigid-foldability**: As demonstrated by Tachi [1], finding a non-trivial flat-foldable configuration of a structure (neither flat nor flat-folded) guarantees rigid-foldability with a single DOF. In the case where a flat-foldable configuration (and therefore rigid-foldable) cannot be found, one can instead characterize the residual strain required when folding the tessellation from its flat to its embedded state by subdividing quads into triangle pairs (effectively increasing the DOFs) and rigid-folding this modified pattern.

- **Flat-foldability**: Unfortunately, no sufficient local condition exists for whether a flat origami pattern is globally flat-foldable, and it is known [3] that the problem is NP-complete. However several necessary local conditions do exist, the most salient of which is Kawasaki’s Theorem [4]: applied to the generalized Miura-ori pattern, it states that if the pattern is flat-foldable, each pair of opposite interior angles around each vertex must sum to \( \pi \). We shall see in Section 4 that in practice, enforcing a loose version Kawasaki’s theorem improves the mechanical performance of the origami.
2 Constructing Generalized Miura-ori tessellations

The inverse Miura-ori design problem can now be formulated: given a smooth surface $M$ in $\mathbb{R}^3$ with boundary that is homeomorphic the disk, an approximation error $\epsilon$, and a length scale $s$, does there exist a generalized Miura-ori tessellation that a) can be isometrically embedded such that the embedding has Hausdorff distance at most $\epsilon$ to $M$; b) has all edge lengths at least $s$? Does there exist such tessellations that satisfy the additional requirement of being flat-foldable?

Experiments suggest that the Gaussian curvature of $M$ significantly influences the difficulty of this inverse problem. Developable surfaces and surfaces with negative Gaussian curvature both readily admit approximations by generalized Miura-ori tessellations; the numerical optimization presented below can also find Miura-ori approximations of positively-curved surfaces, but the space of such tessellations appears to less rich.

2.1 Explicit Construction for Generalized Cylinders

The simplest case is that of generalized cylinders – developable surfaces formed by extruding a planar curve along the perpendicular axis. Therefore, we first give a constructions for approximating generalized cylinders – surfaces $r(s,t) = \gamma(s) + t\hat{z}$ for a plane curve $\gamma$ – by flat-foldable generalized Miura-ori tessellations. We begin by approximating $\gamma(s)$ by a piecewise-linear discrete curve passing through $N$ nodes $\Gamma_i$, and choose a set of $N$ control points $P_i$ on one side of the curve for the Miura-ori structure to pass through. To understand this, consider a strip of paper with uniform width, shown in blue, and rigidly align the left boundary of the strip with the line passing through $\Gamma_1$ and $P_1$ (see Fig. S.1a). Now draw a line (shown dashed) to the next node $\Gamma_2$ and fold the strip along the bisector of $\Gamma_1P_1$ and $P_1\Gamma_2$, shown in red. Continuing this process along all $N$ nodes and control points, with each crease edge given by a bisection yields a construction that has $2N$ free parameters – the position each control point. Then the pattern can be optimized for $\epsilon$ or other design goals such as regularity of the quadrilaterals, etc. and indeed it can be shown that several such strips can be glued together into a generalized Miura-ori pattern approximating a generalized cylinder of any curvature, such as extruded spirals or sine waves that are completely flat-foldable (see Movie1).

Call the previous construction a Miura-ori strip. Given an extrusion parameter $T$, several copies of a Miura-ori strip can be glued into a generalized Miura-ori tessellation approximating the generalized cylinder $r(s,t) = \gamma(s) + t\hat{z}$. Take strip $j$ and displace the right side of the strip by $T$ in the $\hat{z}$ direction, if $j$ is odd, or the left side, if $j$ is even (see Fig.
Figure S.1: **Geometric construction** (a) In-plane strip construction: choose $\Gamma_i$ to discretize a smooth curve $\gamma(s)$, choose control points $P_i$, beginning at $\Gamma_1$ wrap a strip of uniform width (blue) back and forth between the discretization and control points, reflecting over bisectors (red) of the lines through $\Gamma_i$, $P_i$ and $P_{i+1}$, $\Gamma_i+1$. (b) Extrude all points on one side of the strip by $T$. (c) Mirror the strip over the construction plane to produce a single column of Miura-ori cells. (d) Translate and glue copies of the column to create a generalized Miura-ori cylinder.

S.1d), then translate the entire strip rigidly in the $\hat{z}$ direction by $jT$ to complete a new column of Miura-ori cells. It is clear that the strips align as a quadrilateral mesh, that they approximate $r$, and that the faces of the mesh are planar. It remains to be shown that this mesh is developable at the vertices.

Consider $\theta_1$ and $\theta_2$, the interior angles of two consecutive quads in the strip construction, in Fig. S.1b. Because this strip will be mirrored to form a column of Miura-ori cells, developability requires that $\theta_1 + \theta_2 = \pi$. Denoting by $a, b, c$ the lengths of the edges marked in Fig. S.1b, we can lay out a coordinate system with $a(T) = (A, 0, T)$, $b = (B_1, B_2, 0)$ and $c = (C_1, C_2, 0)$ for some $A, B_i, C_i$, and
\[
\cos \theta_1(T) = \frac{AB_1}{\sqrt{A^2 + T^2 \sqrt{B_1^2 + B_2^2}}}
\]
\[
\cos \theta_2(T) = \frac{AC_1}{\sqrt{A^2 + T^2 \sqrt{C_1^2 + C_2^2}}}
\]

Setting \( K(T) = \frac{A}{\sqrt{A^2 + T^2}} \) we have
\[
\cos \theta_1(T) + \cos \theta_2(T) = K(T)\left( \cos \theta_1(0) + \cos \theta_2(0) \right) = 0
\]
since by construction \( \theta_1(0) + \theta_2(0) = \pi \) and so \( \cos \theta_1(0) + \cos \theta_2(0) = 0 \). Therefore \( \theta_1(T) + \theta_2(T) = \pi \) and the tessellation is developable for any \( T \).

Additionally, when consecutive strips of the tessellation are mirrored, the sum of opposite interior angles about any vertex is also \( \theta_1(T) + \theta_2(T) \), and so the construction yields a tessellation that satisfies Kawasaki’s condition (locally flat-foldable) at every node. The tessellation is trivially globally flat-foldable and rigid-foldable, which can be seen by observing that in any folded state the width of each strip in the \( z \) direction is constant and all strips are identical up to rigid translation and reflection (see Fig. S.2).

While our work was under review, we were made aware of a paper that focuses on a small subset of the problems treated here, namely that finding patterns that fit interstitially between two generalized cylindrical surfaces, and by choosing control points \( P \) to fit a second generalized cylindrical surface [2]. Our method provides a simple geometric approach for the surface types solved for numerically in [2]. Our construction recovers this application, but also explicitly guarantees flat- and rigid-foldability, two properties left unproven by the authors of [2]. Because our method guarantees these properties by construction, we implement a simple layout algorithm which directly computes intermediate folding states of the Miura strip using spherical trigonometric relationships between fold and interior angles [5], instead of relying on a numerical simulation to determine these states as in [2].

### 2.2 Curved Surfaces

For surfaces with intrinsic curvature, to our knowledge no explicit generalized Miura-ori construction exists; we propose a numerical optimization algorithm to solve for a tessellation in this setting. Let \( M \) be the target surface that is to be approximated, and parameterize the embedded generalized Miura-ori tessellation by a quadrilateral mesh with vertices \( p_i \). As discussed above, the mesh is generalized Miura-ori if it satisfies a planarity constraint for each face, and a developability constraint at each interior vertex. For a mesh with \( V \) vertices and \( F \approx V \) faces, there are therefore \( 3V \) degrees of freedom and only
Figure S.2: **Global flat-foldability** Starting with the mesh in its designed configuration (some non-trivial folded state), pick a new fold angle with the same MV assignment for the first quad pair in the first column (pink, top left). Using single-vertex fold angle relations from [5] solve for fold angles for each consecutive quad pair in this column (alternating colors along the top) such that the folded width of the pair matches that of the first pair. Note that these fold angles will alternate in MV sign from pair to pair. Because the strip has constant width, the width of the folded column will also be constant through folding and the entire repeated structure will arrive at zero width simultaneously.

$V + F \approx 2V$ constraints, suggesting that the space of embedded Miura-ori tessellations is very rich; it is therefore plausible that one of more such tessellations can be found that well-approximate a given $M$.

Indeed, in practice for many classes of surfaces a tessellation can be found by numerical optimization. The method consists of the following steps:

1. Guess initial positions $p_i^0$ for the vertices of the mesh based on quad mesh parameterization of $M$; this guess closely approximates $M$ but does not necessarily satisfy the planarity, developability or additional constraints.

2. Pin the corners of each unit cell guess to the quads in $M$, ensuring that the generalized Miura-ori surface remains close to $M$.

3. Solve the following constrained optimization problem to produce a developable pattern which approximates $M$. Note that this pinning pattern leaves at least one free node between all fixed nodes in optimization.

$$\min_{p^i} f(p^i, p_0^i) \quad s.t. \quad g_{\text{planarity}}(p^i) = 0, \quad g_{\text{develop}}(p^i) = 0$$

where the objective function $f$ and the constraint functions are described in more detail below.
2.2.1 Initial Positions

The representation of the curved target surface is a regular, orientable quad mesh (all interior nodes have valence four and the normals of the quads are orientable). We will call this the base mesh. The base mesh can be obtained by discretizing the two families of curves formed by a parametrization of the target surface and forms the basis for the initial structure guess provided to the optimization routine. To construct an initial guess for the positions of all nodes in the Miura-ori structure (see Fig. S.3), we proceed by

1. populating each individual quad with 9 nodes (4 at corners, 4 at edges and 1 central),
2. displacing the edge and central nodes to construct a Miura-ori unit cell guess at each quad according to chosen orientations and local length scales, and
3. merging nodes at interior edges by averaging their positions.

Figure S.3: Initial positions (Left) A single base mesh quad (bold) is initially populated with nine nodes (four corner nodes, four edge nodes given by averaging the endpoints and one central node given by averaging the four corners) which will make up a single Miura-ori unit cell (blue). (Middle) The central node and edge nodes in the unit cell (green) are displaced (dashed) according to the choice of pattern orientation to form a structure which “looks” like a single Miura-ori cell. (Right) Because each base mesh quad is converted into a single unit cell independently, we merge nodes (red) between adjacent base mesh quads to form the final mesh. For corner nodes sets (blue) this is only data structures because their positions are fixed, while for edge nodes pairs (green) we also average the two positions to produce the merged node position.

2.2.2 Fixed Nodes

The positions of the four undisplaced corner nodes in each “unit cell” are required to remain fixed throughout optimization. This ensures that the solved structure closely approximates the target surface and further flexibility in designing patterns.
2.2.3 Developability Constraints

The planarity and developability constraints can both be formulated in terms of the vertex positions \( p^i \). For a quadrilateral face with vertices \( p^a, p^b, p^c, p^d \) oriented clockwise, planarity is equivalent to vanishing of the tetrahedral volume

\[
g_{\text{planarity}} = [(p^b - p^a) \times (p^c - p^a)] \cdot (p^d - p^a).
\]

Developability requires that the angles around each interior vertex sum to \( 2\pi \). In other words, if the neighbors of vertex \( i \) are \( n_1, \ldots, n_m \), oriented clockwise, the developability constraint is given by

\[
g_{\text{develop}} = 2\pi - \sum_{j=1}^{m} \angle(p^{n_j} - p^i, p^{n_{j+1}} - p^i)
\]

where the angle between two vectors can be computed robustly using

\[
\angle(v, w) = 2\text{atan2}(\|v \times w\|, \|v\|\|w\| + v \cdot w).
\]

For the numerical optimization, the Jacobians of both constraints are required. Formulas for these derivatives can be readily computed analytically.

2.2.4 Flat-Foldability Constraints

An origami structure is called flat-foldable if it has a folded state in which all of its faces are coplanar (i.e. every face has moved from one plane, the initial paper, to a second plane, the flat-folded state). Consider single flat-folded vertex with four folds. One of the folds will have opposite orientation from the other three. The unique fold can be either of the two folds which do not touch the largest \( \alpha \), and will be tucked inside the other folds in the flat-folded state. In the flat-folded state, consecutive angles interior angles have opposite orientations around the vertex, and walking around this vertex is equivalent to swinging back and forth in the flat-folded state by the \( \alpha \) values. Assuming developability, we know that

\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi.
\]

Because opposite pairs of interior angles share orientation in the flat-folded state, the sums of these pair must be equal (no net change when walking around the entire vertex).

\[
\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4
\]

From these two statements we can see that

\[
\alpha_1 + \alpha_3 = \pi
\]
and

\[ \alpha_2 + \alpha_4 = \pi. \]

In practice, we have found that we cannot satisfy exact flat-foldability on intrinsically curved surfaces. However, we can break open the standard flat-foldability constraints into inequalities which express bounds on a flat-foldability residual. Notice that we have a single scalar at each interior vertex which represents the flat-foldability residual.

\[ r_{ff} = \pi - (\alpha_1 + \alpha_3) = -\left(\pi - (\alpha_2 + \alpha_4)\right) \]

Introducing a tolerance \( \epsilon \) on \( r_{ff} \) in the form of a pair of inequality constraints allows the each pair of alternating angles at an interior vertex to sum to a value within \( \epsilon \) of \( \pi \).

\[ g_{\text{flat-foldability}}(p^i) = \pm r_{ff} - \epsilon \leq 0 \]

In the limit \( \epsilon \to 0 \) these inequalities reduce to the standard equality Kawasaki condition.

### 2.2.5 Special Cases

- **Rotational Symmetry**

  For surfaces with rotational symmetry we enforce developability constraints over a symmetric strip using periodic boundary conditions. The symmetric strip can then be used to reconstruct the full developable Miura-ori structure. This strategy is particularly useful when analyzing the asymptotic behavior of solved patterns over magnitudes of order changes in pattern resolution, as the computational demands are linear in strip resolution but the size of the solved pattern grows quadratically with strip resolution. We employed this strategy for the sphere, hyperboloid and all mixed curvature examples.

- **Triangulated Pattern**

  For some examples, the developability constraint residuals fail to vanish completely. Typically these non-zero values are on the order of at most \( 1e^{-6} \). These residuals can still introduce error in the layout process, however, so in these cases we employ a second phase of optimization:

  - introduce additional degrees of freedom in the optimization by dropping the quad planarity constraint,
  - triangulate the pattern so that each interior node has six incident edges (and therefore six incident interior angles) and
  - solve \( g_{\text{develop}} = 0 \) over six angles rather than four at each interior node.
We only found need to employ triangulation on surfaces with rotational symmetry, and we report the relevant residuals associated with both optimization phases with each example.

- Normalized Quad Planarity

Because the quad planarity constraint $g_{\text{planarity}}(p^i) = 0$ is just the volume of the quad, it scales as $L^3$ with the length $L$ of the pattern edges. For most examples we are able to solve these constraints to arbitrary precision and the scaling is irrelevant. However, for the hyperboloid we compute patterns over two orders of magnitude of pattern resolution, so the scaling of $g_{\text{planarity}}$ becomes relevant: more highly resolved patterns can more easily satisfy quad planarity by virtue of their smaller length scales. To address this, we solve a normalized version of quad planarity:

$$g_{\text{planarity-norm}} = \frac{g_{\text{planarity}}}{L_j^3} = 0,$$

where $L_j$ is a length scale associated with the initial geometry of the $j^{th}$ quad. We choose $L_j$ to be the mean of the four initial side lengths of quad $j$.

### 2.2.6 Objective Function

The objective function minimizes changes in the lengths of pattern edges and cross edges (see Fig. fig:objective) of the initial guess. Edge $i$ with current length $L^i$ and initial length $L_0^i$ contributes

$$E_i = \frac{1}{2L_0^i}(L^i - L_0^i)^2.$$ 

Because this energy is not balanced against other terms we neglect a stiffness prefactor. The objective function is zero at the beginning of each run and $\sum_{i=1}^{M} E_i$ for a structure of $M$ total edges (pattern and cross) thereafter. The purpose of the objective function is to preserve the initial user-provided positions as closely as possible during optimization ($E_i$ has no physical significance).

### 2.2.7 Numerical Optimization Approach

We implement the numerical optimization in Matlab using the Interior Point algorithm of `fmincon`. Fixed nodes can be implemented either as linear (which require no Jacobian) or simply by leaving these variables out of $p^i$. We provide analytic Jacobians for planarity and developability constraints (non-linear equality) and flat-foldability constraints (non-linear inequality). Successful optimizations typically find minima and satisfy constraints by a maximum residual of 1e-10 within several hundred iterations.
Figure S.4: **Constraint patterns** Blue nodes: free, red nodes: fixed, dashed quads: $g_{\text{planarity}}$, open circles: $g_{\text{develop}}$ and $g_{\text{flat-foldability}}$, green arcs: rotational symmetry pairs

3 Examples

See Fig. S.6 for additional structures and patterns not presented in the main text.

4 Foldability

Note that satisfying $g_{\text{planarity}} = 0$ and $g_{\text{develop}} = 0$ guarantees the existence of only two states (three counting the mirror symmetric configuration obtained by flipping all MV assignments) of the curved Miura-ori structure: a single folded configuration in $R^3$ and a developed pattern in $R^2$. The existence of other folded states of the pattern and, in particular, the existence of a continuous, isometric global motion from flat to solved states (i.e. a rigid folding) are also of interest. The existence of a rigid folding of a quad-based
The objective function is based on linear springs at the pattern edges (solid) and cross edges of each quad (dashed) in the initial configuration.

generalized Miura-ori structure would necessarily have a single DOF and would therefore constitute a mechanism, an obviously desirable property for engineering applications.

Tachi [1] finds that a generic quadrilateral structure is rigid-foldable if it is

- everywhere locally flat-foldable (satisfies the Kawasaki condition) and
- a non-trivial configuration (neither flat nor flat-folded states) of the structure exists.

These are sufficient conditions for the existence of a rigid folding motion from flat to flat-folded, passing through the non-trivial configuration. This means that if we can solve for a folded state of a curved Miura-ori structure with flat-foldability enforced exactly at all interior nodes, we are guaranteed a rigid-foldable structure with one DOF. Such a structure would be able to fold from flat to its solved state (non-trivial configuration) and past its solved state to a flat-folded state (all faces are coplanar and all fold angles are \(\pm \pi\)).

All generalized cylinders examples we produce are flat-foldable and therefore rigid-foldable by geometric construction. In the case of generic surfaces, however, we are unable to find exactly flat-foldable solutions. In order to fold generic material structures then, we expect geometric frustration to induce bending in quad faces in intermediate folding states. We characterize the geometric frustration in the folding process with a simple mechanical simulation, and show that even if an exactly flat-foldable structure cannot be found, optimizing with bounds on the flat-foldability residual mitigates this frustration. Recall the inequality constraint \(g_{\text{flat-foldability}}\) from Section 2.2.4. Because of the relationship between flat-foldability and rigid-foldability laid out in [1], we expect that as we tighten the \(\epsilon\) bounds on \(g_{\text{flat-foldability}}\) the solved structure approaches rigid-foldability as well.

Keep in mind that the structures discussed so far in this section are assumed to be quad-based with all valence 4 interior nodes, and that rigid-foldability would preserve the planarity of quads between flat and folded states. Instead we divide each quad into two triangles (effectively dropping quad planarity and adding extra DOFs to the structure) and in practice are able to rigidly fold these subdivided structures from flat to solved (folded) states by allowing each quad to bend along the newly introduced crease in intermediate
folding states. This folding motion is a rigid folding, but does not constitute a mechanism because of the additional DOFs. We compute these rigid folding motions using a simple mechanical simulation detailed below.

### 4.1 Simulation Method

Using the hyperbolic paraboloid pattern (hypar), we begin by choosing a single fold near the center of the pattern (see Fig. S.7). This fold is then constrained to incrementally changing fold angles from solved to flat in simulation, the actuation of which propagates throughout the structure by the equilibration of bending energies in the quads, effectively unfolding the pattern mechanically. All edge lengths remain constant (enforced by non-linear constants) during simulation, and thus the computed folding motion is rigid.

Stating this procedure formally, we solve the following optimization problem

\[
\min_{p_i} f(p_i, p_i^0) \quad \text{s.t.} \quad g_{\text{edges}}(p_i) = 0, \ g_{\text{fold}}(p_i) = 0,
\]

where

\[
f_j(p_i, p_i^0) = \frac{1}{2} k_j \theta_j^2
\]
is the sum of all bending energies in the quad faces,

\[ g_{\text{edges}}(p^i) = \|e_k\| - L_k \]

is the edge length constraint, and is enforced at all edges with initial lengths \( L_k \) in the triangulated pattern, and

\[ g_{\text{fold}}(p^i) = \theta - \theta_{\text{pinch}} \]

is the pinched fold angle constraint, enforced at a single fold in the interior of the pattern with \( \theta \) its fold angle and \( \theta_{\text{pinch}} \) the prescribed fold angle. Each incremental optimization takes the equilibrium node positions at the previous intermediate folding configuration as \( p_i^0 \).

Note that the only bending energies present in \( f \) are all within the quad faces. No fold angle, which resides at an interior edge between two adjacent quads, contributes to the objective function. And with the exception of the pinched fold, all fold angles are unconstrained and can move freely during optimization. Therefore, if the quad-based Miura-ori structures we solve for were indeed rigid-foldable without additional DOFs from triangulation, we would expect to find a zero-energy configuration of the mesh at every intermediate state between flat and folded. Taken together these configurations would constitute a rigid folding of the quad mesh. We do not, however, observe such intermediate states in any folding simulations and therefore conclude that these structures can only be rigidly folded with the additional DOFs.

Figure S.7: **Triangulation of quad-based hypar pattern** Solid lines: original patterns quads, dashed lines: Delaunay subdivision of quads, red lines: “pinched” fold
4.2 Structural Mechanics of Origami

To compare our simulation results with real material structures, we connect the bending stiffnesses assigned in simulation to the Young’s modulus and bending stiffness of the material/structure.

Our bending model is based on adjacent triangles in each flat quad, so we need to connect the folding of a triangle pair to the uniform bending of a linearly elastic material piece of the same area and thickness.

Consider a triangle pair with areas $A_1$ and $A_2$ and shared edge length $L$. This pair has the same area as a rectangle of width $w = L/2$ and length $a = 2(A_1 + A_2)/L$. If we bend this rectangle uniformly along its length into a circular arc also of length $a$ (see Fig. S.8a and Fig. S.8b), we observe that the radius of curvature of this arc is $R = a/\theta$, where $\theta$ is the fold angle (i.e. exterior to the dihedral angle between the two faces). This comes from the fact that $\alpha/2 + (\pi - \theta)/2 + \pi/2 = \pi$.

![Figure S.8](image)

**Figure S.8**: Bending stiffness (a) Bending stiffness triangle pair with inscribed arc (b) Profile of bent triangle pair

Now that we can connect the geometry of bending of two triangles and a rectangular volume, we can derive a bending stiffness by equating the bending energies.

A uniformly bent sheet with length $a$, constant thickness $h$, second moment of inertia $I$ and Young’s modulus $E$ has strain energy due to stress along its length

$$U^\theta_b = \frac{1}{2} EI\kappa^2 a,$$
where $\kappa$ is the curvature of the sheet’s mid-plane. We can compute $I = \int_A z^2 dA$ for the bent sheet where $A$ is the cross-sectional area, $z$ is in the direction of the thickness and $L/2$ is the width.

$$I = \frac{1}{24}Lh^3$$

Substituting $I$ and $\kappa = 1/R$ gives

$$U_b = \frac{1}{48}\frac{ELh^3a}{R^2}.$$ 

Equating this to the discrete bending energy model $f_j$ above gives

$$\frac{1}{2}k_j\theta^2 = \frac{1}{48}\frac{ELh^3a}{R^2},$$

where all parameters now belong to the two triangles inside quad $j$. Substituting $R = a/\theta$ gives

$$\frac{1}{2}k_j\theta^2 = \frac{1}{48}\frac{ELh^3a}{a}\theta^2.$$ 

Substituting $a = 2(A_1 + A_2)/L$ gives our final bending stiffness $k$.

$$k_j = \frac{1}{48}\left(\frac{L^2}{A_1 + A_2}\right)Eh^3$$

For results presented in the main text we use $E = 10^9\text{N/m}^2$ and $h = 10^{-4}\text{m}$, reasonable values of paper-like material, to compute $k_j$ and we non-dimensionalize the total bending energies by the largest observed bending energy in a single material quad across all simulations, $9.764 \times 10^{-8}\text{J}$.

### 4.3 Experimental measurement of Hypar stiffness

As discussed earlier, our simulations show that a larger flat-foldability residual leads to a higher energetic barrier between the flat and folded configurations. This bistability is likely a desired property in deployable structures that need to be (at least) locally stable. To verify this trend experimentally, we measure the stiffness of a pair of calculated hypars with different flat-foldability residuals. After laser-cutting the tessellations onto a sheet of paper, we fold these structures and attach inextensible thread and paper paddles to one unit cell close to the boundary of the folded structure. We then conduct a simple force extension experiment using an Instron (see Fig. S.9) over a strain range of 0.2 using the following protocol: extend the structure at 5mm/s until the maximum nominal strain is reached, and then reverse the process till the force goes back to zero. We then repeat the experiment two more times. We find that the first “run-in” experiment is different
and reflects the irreversible deformations associated with the virgin origami structure, but eventually the force-extension plot settles onto a steady curve. We see that the curve for the hypar with the larger flat-foldability residual is stiffer, and underscores the change in global mechanical response of the structure by a modification of local geometry, as predicted by our simulations.

Figure S.9: **Stiffness experiment** (a) Structures corresponding to patterns $\epsilon_{ff} = \epsilon_0, \epsilon_0/10$ (b) Loading a hypar in the Instron

5 Accuracy vs. Effort: Hyperboloid of a Single Sheet

In addition to providing examples of origami surfaces with a variety of curvatures, we are also interested in optimizing the trade-off between approximation accuracy and pattern resolution. It is natural to expect that as we increase the resolution of generalized Miura-ori surface, we would be able to approximate its target surface more accurately. However, it is also easy to imagine a scenario, in particular in real-world applications, in which increased resolution incurs some fabrication cost (time and complexity). It is also unknown whether significantly increasing resolution and accuracy would incur an additional material cost, i.e. the limiting behavior of the areas of increasingly resolved generalized Miura-ori surfaces. We use simple numerical experiments fitting the hyperboloid of one sheet to provide insight into these questions, illustrating the trade-off between accuracy and resolution.

The hyperboloid of one sheet has a number of properties that make it a natural setting for investigating these questions computationally.
• Negative Gauss curvature: As we have observed, negatively curved surfaces are more natural settings for fitting generalized Miura-ori surfaces. We expect fast, accurate convergence on the hyperboloid without having to resort to optimization setups with additional DOFs.

• Rotational symmetry: We can reduce the entire surface to a single symmetric strip, which significantly reduces the computational demands of increased surface resolution in optimization. In particular, the size of the dense Jacobian provided to \texttt{fmincon} is quadratic in the number of unit cells per symmetric strip, rather than quartic, which would be the case without rotational symmetry.

• Ruled surface: Conveniently, the hyperboloid has two symmetric families of rulings. Taken together, these families form a natural base mesh for optimization, so the choice of symmetric strip is not arbitrary, but rather given by the geometry of the hyperboloid and the desired resolution.

5.1 Base Mesh: Diagonal, Rotationally Symmetric Strip

A hyperboloid of one sheet with waist radius $a$ and rotational symmetry about the $z$-axis is given implicitly by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1.$$ 

Choosing $a = \sqrt{2}/2$ and $c = 1$, simply for aesthetics, this surface can be parameterized by

$$x(t, v) = \cos t + v(\pm \sin t - \cos t)$$
$$y(t, v) = \sin t + v(\mp \cos t - \sin t)$$
$$z(t, v) = 2(v - \frac{1}{2}).$$

We will focus on the surface patch given by $t \in [0, 2\pi), v \in [0, 1]$. The sign change in the parameterization gives two families of rulings (see Fig. S.10). A single ruling, which runs diagonally on the surface of the hyperboloid, can be obtained by holding $t$ constant and varying $v$. This parameterization is convenient because at $v = 0$ we have the bottom circular boundary of the surface patch of interest.

By sampling the rulings families over an even number of uniform intervals along the bottom circle we can construct the diagonal grid seen in Fig. S.10. Furthermore, if we divide the bottom circle into $2(n + 1)$ arcs and extend rulings from the endpoints of these arcs, each diagonal, rotationally symmetric strip in the base mesh will have $n$ quads (giving
Figure S.10: **Hyperboloid of one sheet** The two families of rulings (black) intersect to form a natural base mesh. Here we have chosen 10 quads in each diagonal, rotationally symmetric strip. Two consecutive rulings (bold black) mark the periodic boundaries of a symmetric strip.

$2n(n+1)$ quads over the whole hyperboloid). The symmetry between rulings families also guarantees that the left and right nodes in each quad are themselves rotationally symmetric. Recall that all of the nodes in the base mesh remain fixed during optimization, allowing us to exploit the underlying symmetry of the base mesh via the constraint pattern in Fig. S.4b.

### 5.2 Examples

We produce numerical results for hyperboloids with 10 to 100 (intervals of 10) and 100 to 1000 (intervals of 100) unit cells per symmetric strip for a total of 19 generalized Miura-ori structures over two orders of magnitude in strip resolution (see Fig. S.11 and Fig. S.12). We use diagonally-symmetric developability constraints and area normalization in quad constraints (see Section 2.2.5) to ensure scale-independent satisfaction of convergence tolerances at small length scales ($g_{\text{planarity-norm}} = 0$ and $g_{\text{develop}} = 0$ are both satisfied within tolerances of $10^{-10}$).
Figure S.11: **Generalized Miura-ori hyperboloids (Left to Right)** 10, 100 and 1000 unit cells per symmetric strip

Figure S.12: **Generalized Miura-ori hyperboloid development** 10 unit cells per symmetric strip

### 5.3 Computing the Hausdorff Distance

The Hausdorff distance $d_H$ is defined as the maximal distance between the points in one set and their closest points in another set, as viewed from both sets. More formally, for two sets $M$ and $S$, $d_H$ is given by

$$d_H(M, S) = \max[d(M, S), d(S, M)]$$

$$d(M, S) = \max[d(x, B)], \ \forall x \in M$$

$$d(x, S) = \min[d(x, y)], \ \forall y \in S.$$ 

Denoting the Miura-ori hyperboloid $M$ and the target hyperboloid $S$, we compute $d(M, S)$ between each Miura-ori hyperboloid and the target surface computationally, as no analytic
expression of distance from a point in space to the hyperboloid surface exists, and set this equal to \( d_H(M, S) \). Because the target hyperboloid is a continuous surface consisting of infinite points we cannot compute \( d(S, M) \), but we note that in this particular case \( d_H(M, S) = d(M, S) \), up to some error bound, as proved next.

Let \( M \) be a quadrilateral mesh (possibly non-developable with non-planar faces) and \( S \) a compact smooth Riemmanian manifold (possibly with boundary) embedded in \( \mathbb{R}^3 \). For each vertex \( v_i \) of \( M \), let \( \tilde{v}_i \) be its orthogonal projection onto \( S \) (we assume that all points of \( M \) are close enough to \( S \), relative to the curvature of \( S \), so that their projections are unique). Let \( \delta = \max \|v_i - \tilde{v}_i\| \) and

\[
\epsilon = \max_{p \in S} \min_i g(p, \tilde{v}_i)
\]

where \( g(p, q) \) is geodesic distance on \( S \); in other words, \( \epsilon^{-1} \) bounds how densely the projected mesh points sample the surface. Finally, let

\[
\eta = \max_{i \sim j} g(\tilde{v}_i, \tilde{v}_j),
\]

where the maximum is taken over all projections of neighboring vertices on \( M \). Then the Hausdorff distance between \( S \) and \( M \) satisfies

\[
d_H(S, M) \leq 2\delta + \max(\eta/2, \epsilon).
\]

First, notice that if \( v_i \) and \( v_j \) are neighboring vertices, \( \|v_i - v_j\| \leq \eta + 2\delta \). Let \( p \) be a point on \( M \). By the triangle inequality, if \( v \) is the closest vertex of \( M \) to \( p \), then \( \|p - v\| \leq \eta/2 + \delta \), and \( \|p - \tilde{v}\| \leq \eta/2 + 2\delta \), so

\[
d(M, S) \leq \eta/2 + 2\delta.
\]

Next, clearly \( d(S, M) \leq \epsilon + \delta \), proving the theorem.

Notice that displacing the vertex \( v_i \) in the direction normal to the surface changes \( \delta \), but not the other bounds, therefore finding such normal displacements that minimize \( \delta \) also minimizes the above bound on Hausdorff distance. When the points \( v_i \) are allowed to slide tangentially (which may be required in order to enforce the Miura constraints on \( M \)) minimizing \( \delta \) remains a good heuristic, as for example when fixing some of the points at \( v_i = \tilde{v}_i \) to bound increases in \( \eta \) and \( \epsilon \).

To compute \( d(M, S) \) for the hyperboloid, consider a point

\[
p = (x_p, y_p, z_p)
\]

in \( \mathbb{R}^3 \) and a surface parameterization

\[
S(t, v) = (x_s(t, v), y_s(t, v), z_s(t, v)).
\]
The distance $D$ between $p$ and a point on $S$ is given by

$$D(t, v) = \sqrt{(x_p - x_s(t, v))^2 + (y_p - y_s(t, v))^2 + (z_p - z_s(t, v))^2}.$$ 

For each point in the generalized Miura-ori hyperboloid we can identify its closest point on the target hyperboloid $S$ by minimizing $D^2$ with respect to $t$ and $v$, which we implement by providing analytic Jacobians to Matlab’s \texttt{fminunc}. Computing $d_H$ for each optimization result is straightforward once these correspondences are established.

5.4 Accuracy/Effort Trade-Off

We construct a cost function from weighted, linear combinations of the data sets $d_H$ (Hausdorff distance between Miura-ori and target hyperboloids) and $N$ (total number of unit cells in the Miura-ori hyperboloid). We normalize each set by its largest value to produce

$$\hat{d}_H = \frac{d_H}{\|d_H\|_{\infty}} \quad \text{and} \quad \hat{N} = \frac{N}{\|N\|_{\infty}}.$$ 

The cost function $C$ is a weighted sum of $\hat{d}_H$ and $\hat{N}$ (weights $w_d$ and $w_N$, respectively).

$$C = w_d \hat{d}_H + w_N \hat{N}$$

By tuning the ratio $w_N/w_d$ we can produce cost functions with different minima and therefore different optimal Miura-ori hyperboloids.

5.5 Area Convergence

Because all quads in the Miura-ori hyperboloids are planar, we simply sum their areas to compute the total area of a structure. These areas converge in the limit of strip resolution $n \to \infty$ and the area asymptote $A_0$ for the Miura-ori hyperboloids is $\sim 24.13$, whereas the actual area of the smooth hyperbolic target patch is $\sim 10.77$. This factor of $\sim 2.24$ difference could be likely be reduced with different initial position parameters, but we expect any reduction to be minimal. Our convergent Miura-ori approximation constitutes an isometric \textit{wrapping} of the hyperboloid as defined in [6].

For comparison, we can construct an alternative developable approximation of the same hyperboloid using a single family of rulings as shown in Fig. S.13 and Fig.S.14. In this construction, a symmetric strip consists of two triangles generated by two consecutive
rulings and a diagonal between them. To first order in $t$, the area of these two triangles (a symmetric strip), is

$$t\sqrt{5 - t + \frac{5}{4}t^2}, t = \frac{2\pi}{2(n + 1)}.$$ 

Again, $n$ is the number of quads in a single symmetric strip and $2(n+1)$ is the total number of strips, borrowed from the Miura-ori construction for comparison. From this it is easy to see that the alternative construction has an total area approaching $2\pi\sqrt{5} \approx 14.05$ in the limit $n \to \infty$, for a ratio of $\sim 1.30$. While this singly-corrugated construction has a limiting area which more closely approximates the hyperboloid, the convergence of this area still follows the length scale of the discretization and such specialized constructions only exist for special target surfaces, such as ruled surfaces. Future work could classify these limiting area ratios for different origami tessellations and different surface types.

Figure S.13: **Hyperboloid of one sheet, alternative construction** Using a single family of rulings and with pairs of triangles between consecutive rulings to generate a developable construction of the hyperboloid.
Figure S.14: Hyperboloid of one sheet, alternative construction, development

References


