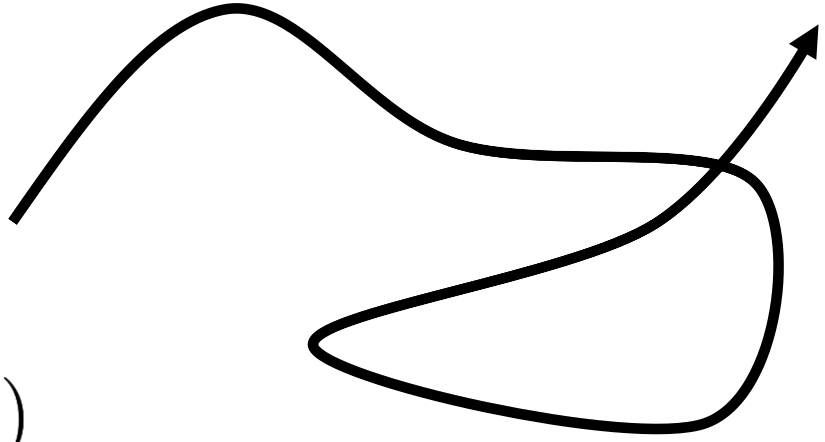
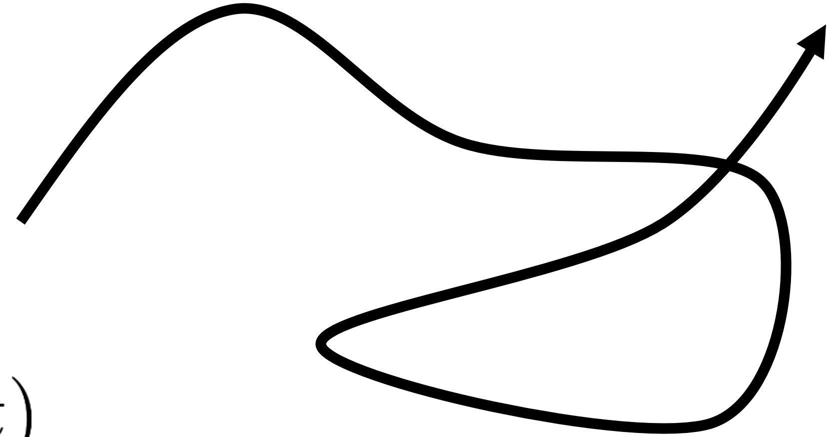
Curves and Splines

Curves in Spaces



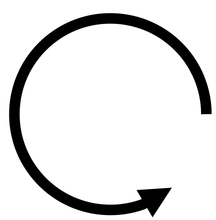
Parametric function $\gamma(t)$

Curves in Spaces



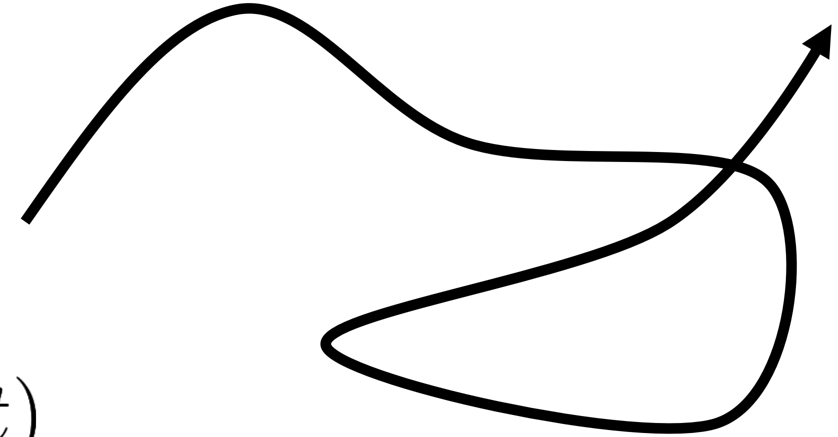
Parametric function $\gamma(t)$

Simple curves have well-known formulas:



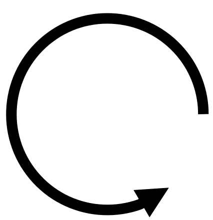
$$\gamma(t) = (\cos t, \sin t)$$

Curves in Spaces



Parametric function $\gamma(t)$

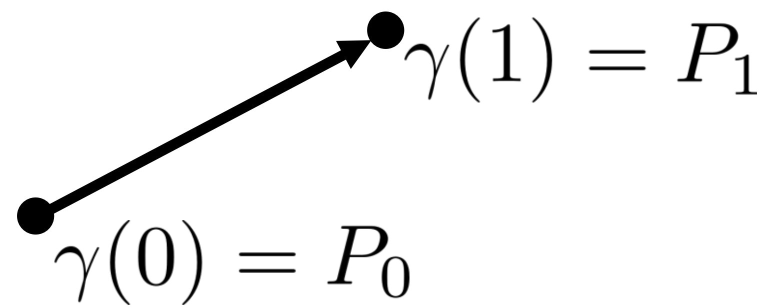
Simple curves have well-known formulas:

 $$\gamma(t) = (\cos t, \sin t)$$

What to do in general?

Linear Interpolation

Straight line segment between two points



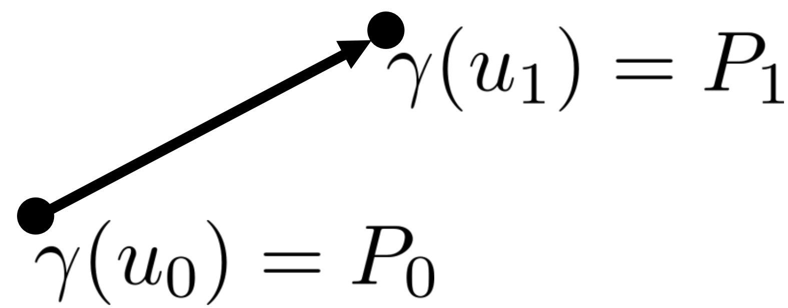
Linear Interpolation

Straight line segment between two points

$$\gamma(t) = P_0 + t(P_1 - P_0)$$
$$\gamma(t) = (1-t)P_0 + tP_1$$

Linear Interpolation

Straight line segment between two points

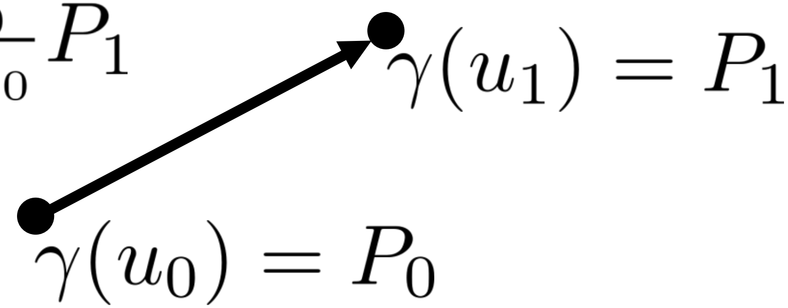


Also works for arbitrary parameterization

Linear Interpolation

Straight line segment between two points

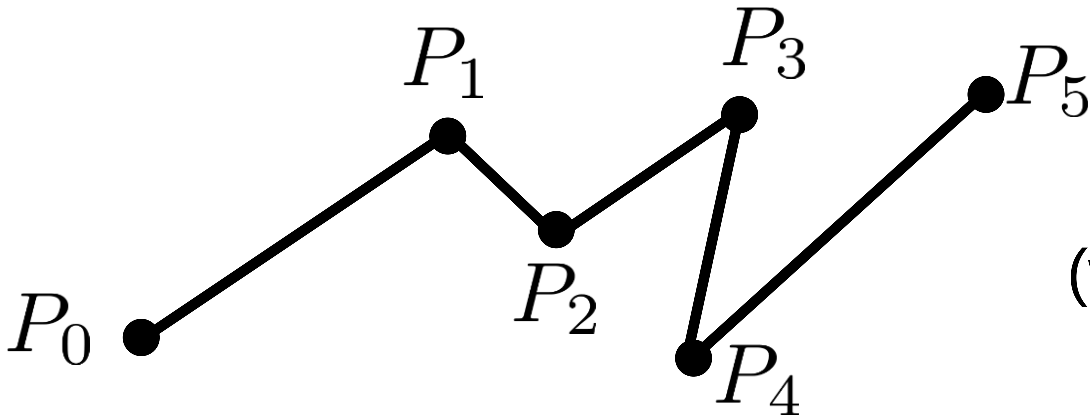
$$\gamma(t) = \frac{u_1 - t}{u_1 - u_0} P_0 + \frac{t - u_0}{u_1 - u_0} P_1$$



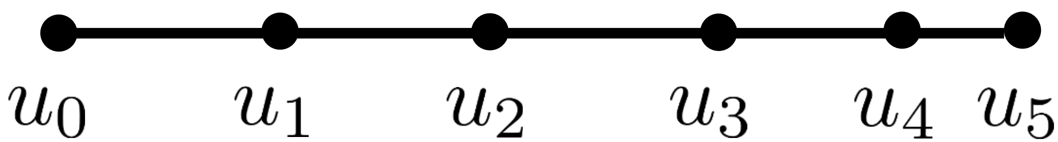
Also works for arbitrary parameterization

Piecewise Linear Interpolation

Straight line segment between point list



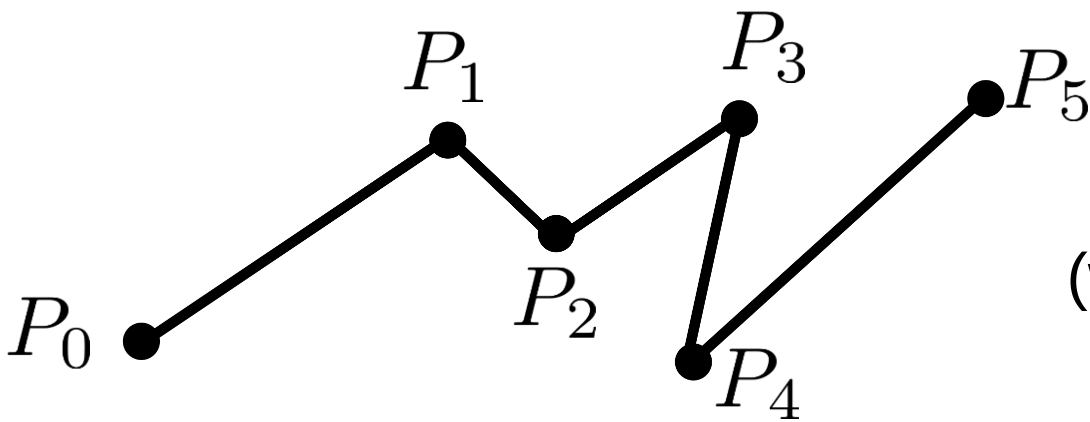
points in space
(where curves goes)



points in parameter space
(**knots**)
(how fast it goes)

Piecewise Linear Interpolation

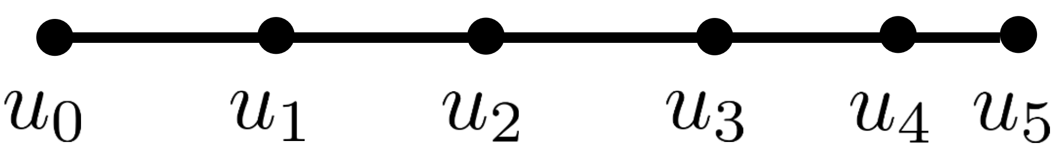
Straight line segment between point list



points in space
(where curves goes)

$$\gamma(t) = \frac{u_{i+1} - t}{u_{i+1} - u_i} P_i + \frac{t - u_i}{u_{i+1} - u_i} P_{i+1}$$

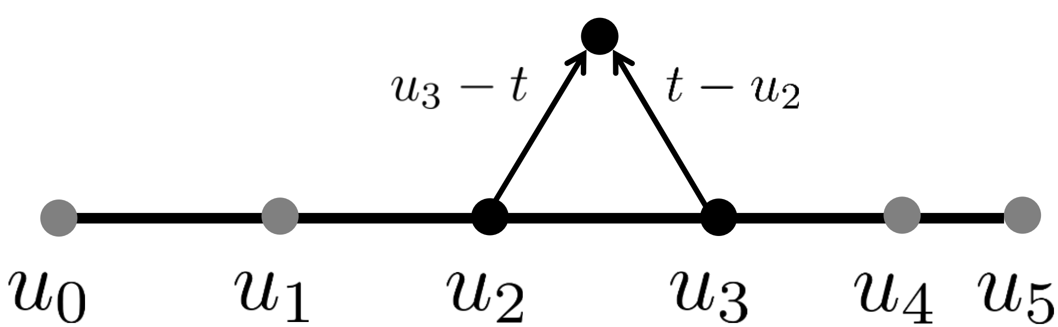
points in parameter space
(**knots**)
(how fast it goes)



Piecewise Linear Interpolation

"Pyramid Notation"

$$\gamma(t) = \frac{u_{i+1} - t}{u_{i+1} - u_i} P_i + \frac{t - u_i}{u_{i+1} - u_i} P_{i+1}$$

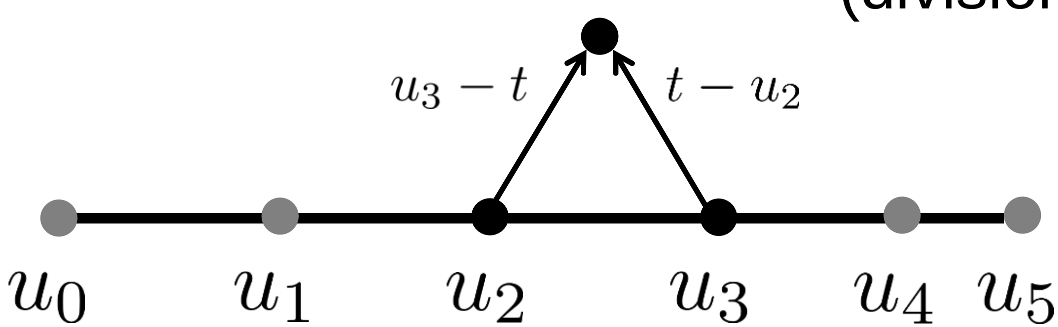


Piecewise Linear Interpolation

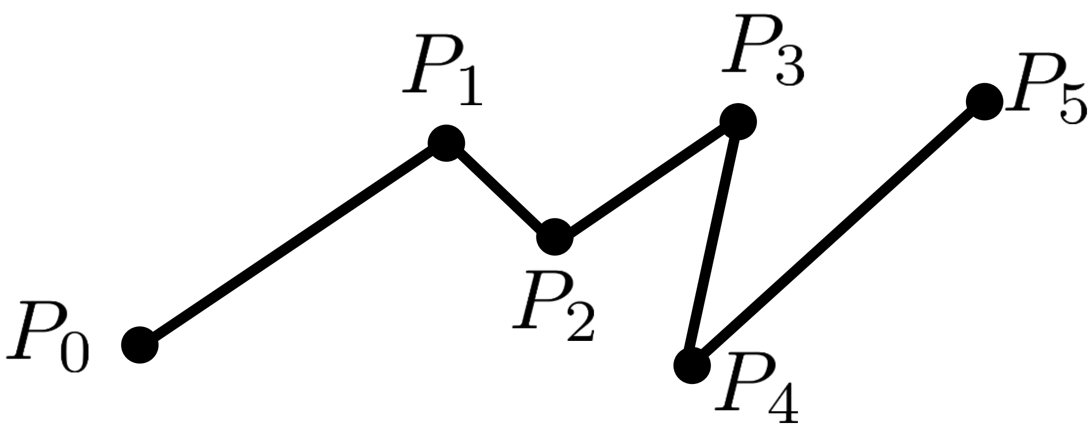
"Pyramid Notation"

$$\gamma(t) = \frac{u_{i+1}-t}{u_{i+1}-u_i} P_i + \frac{t-u_i}{u_{i+1}-u_i} P_{i+1}$$

(division by **sum** implicit)

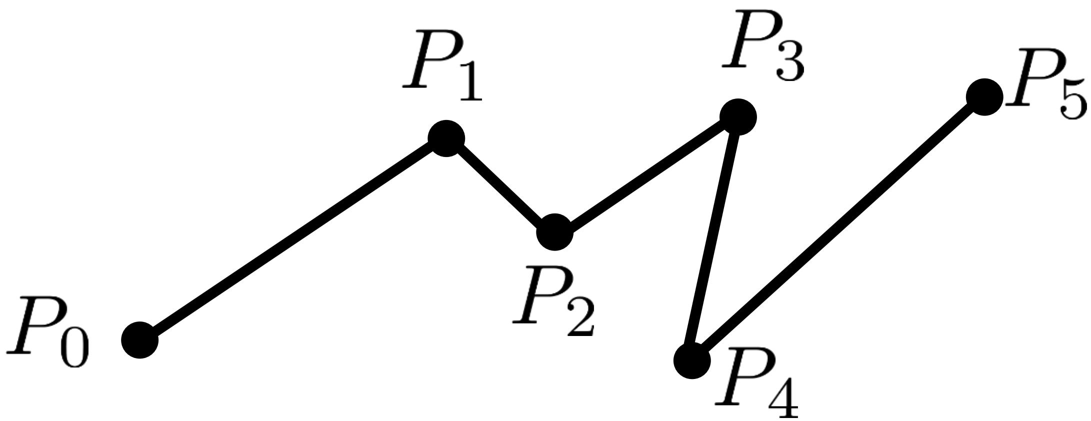


Piecewise Linear Interpolation



Easy, but “chunky” – only C^0

Piecewise Linear Interpolation



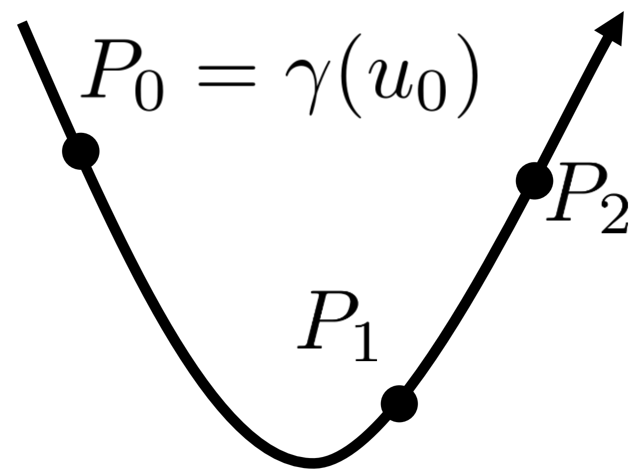
Easy, but “chunky” – only C^0

Continuity notation: C^n means continuous after taking n derivatives

Lagrange Interpolation

Given some points, find polynomial

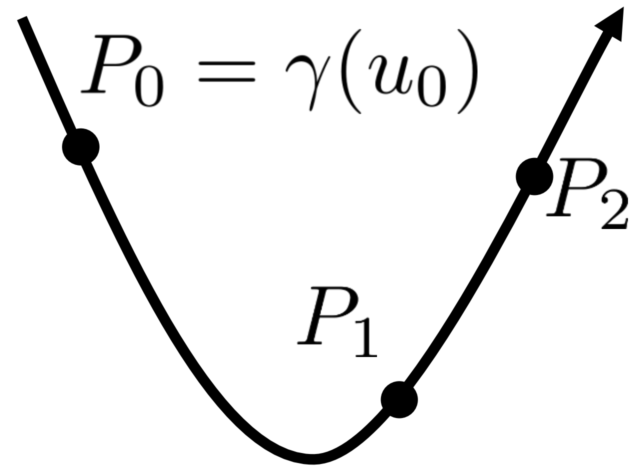
$$\gamma(t) = \left[\begin{array}{c} a_x + b_x t + c_x t^2 + \ldots \\ a_y + b_y t + c_y t^2 + \ldots \end{array} \right]$$



Lagrange Interpolation

Given some points, find polynomial

$$\gamma(t) = \begin{bmatrix} a_x + b_x t + c_x t^2 + \dots \\ a_y + b_y t + c_y t^2 + \dots \end{bmatrix}$$

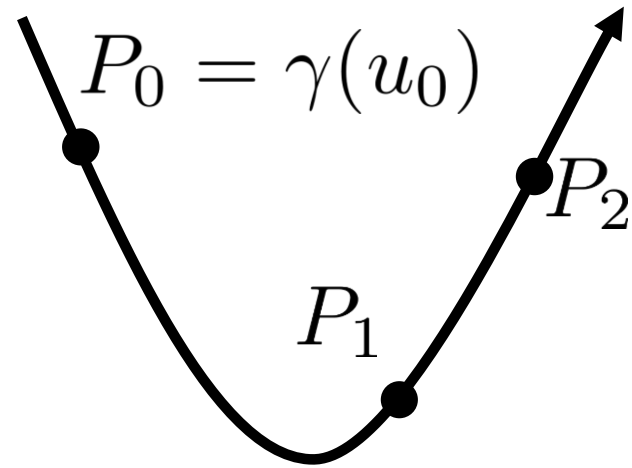


Notice: each coordinate is **linear combination** of a power of t

Lagrange Interpolation

Given some points, find polynomial

$$\gamma(t) = \begin{bmatrix} a_x & b_x & c_x & \cdots \\ a_y & b_y & c_y & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \end{bmatrix}$$

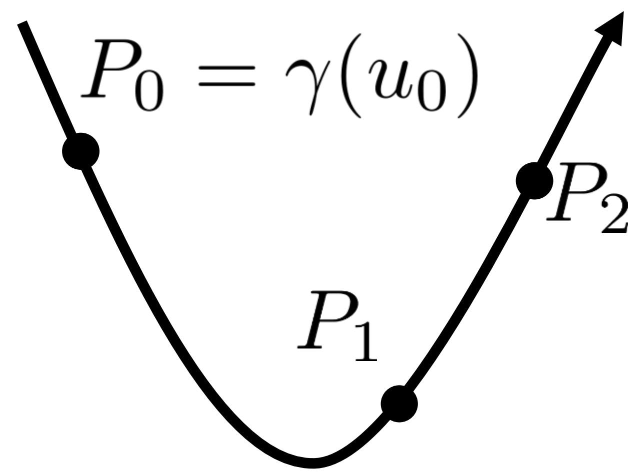


Notice: each coordinate is **linear combination** of a power of t

Lagrange Interpolation

Given some points, find polynomial

$$\gamma(t) = C_{2\times k} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \end{bmatrix}_{k\times 1}$$



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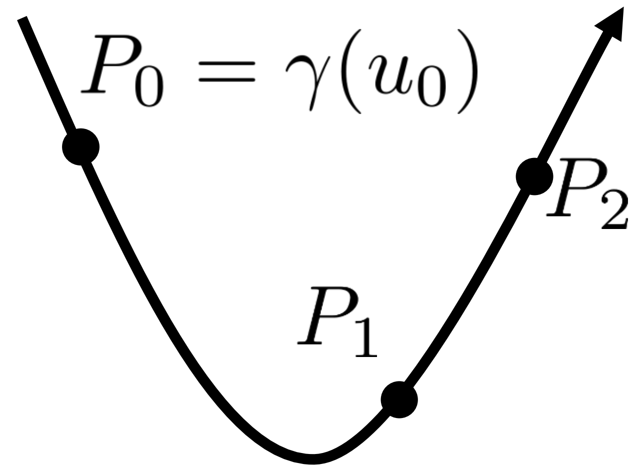
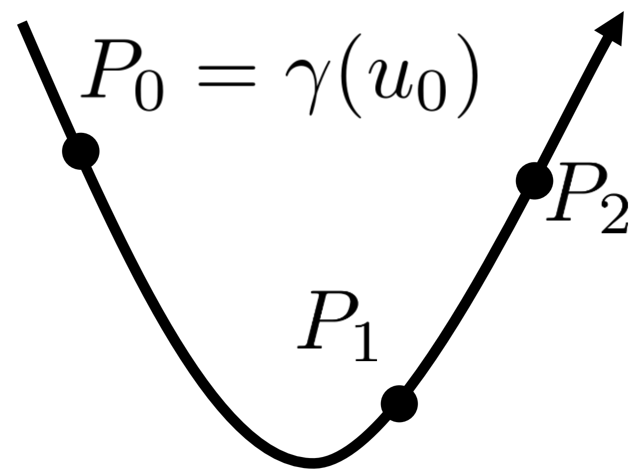


How to pick k?

Lagrange Interpolation

Given some points, find polynomial

$$P_i = C_{2\times k} \begin{bmatrix} 1 \\ u_i \\ u_i^2 \\ \vdots \end{bmatrix}_{k\times 1}$$

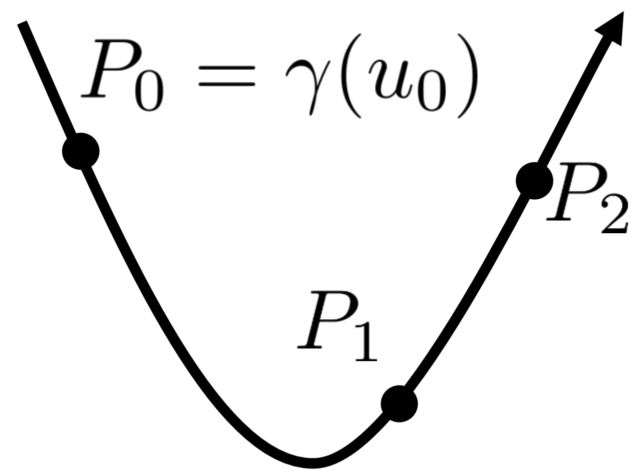


How to pick k? Use known points

Lagrange Interpolation

Given some points, find polynomial

$$\left[\begin{array}{c|c|c} P_0 & P_1 & P_2 \end{array}\right] = C_{2\times k} \begin{bmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ u_0^2 & u_1^2 & u_2^2 \\ \vdots & \vdots & \vdots \end{bmatrix}_{k\times n}$$



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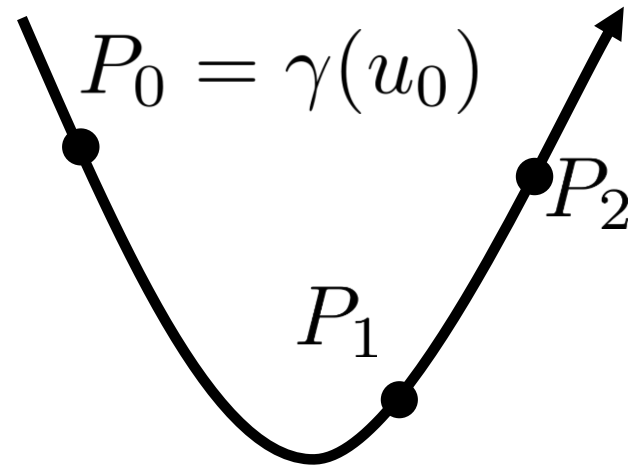
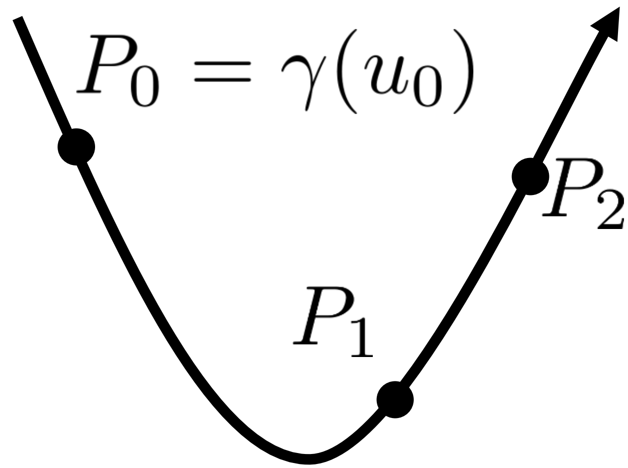


How to pick k? Use k = n

Lagrange Interpolation

Given some points, find polynomial

$$\left[\begin{array}{c|c|c} P_0 & P_1 & P_2 \end{array}\right] = C_{2\times n} \begin{bmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ u_0^2 & u_1^2 & u_2^2 \end{bmatrix}_{n\times n}$$

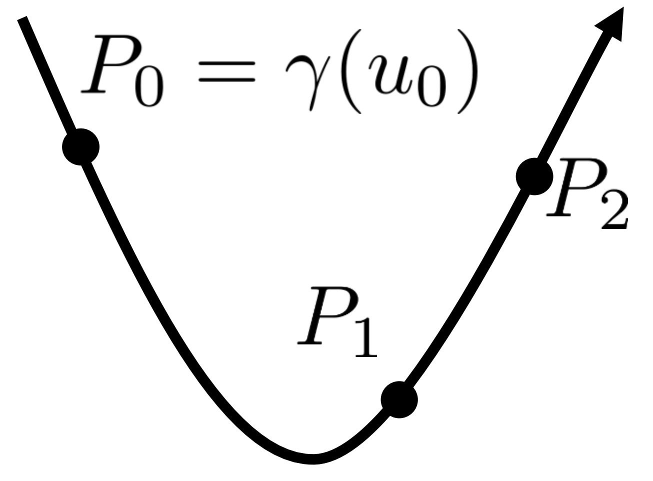


How to pick k? Use k = n

Lagrange Interpolation

Given some points, find polynomial

$$C_{2\times n} = \left[\begin{array}{c|c|c} P_0 & P_1 & P_2 \end{array}\right] \begin{bmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ u_0^2 & u_1^2 & u_2^2 \end{bmatrix}^{-1}$$



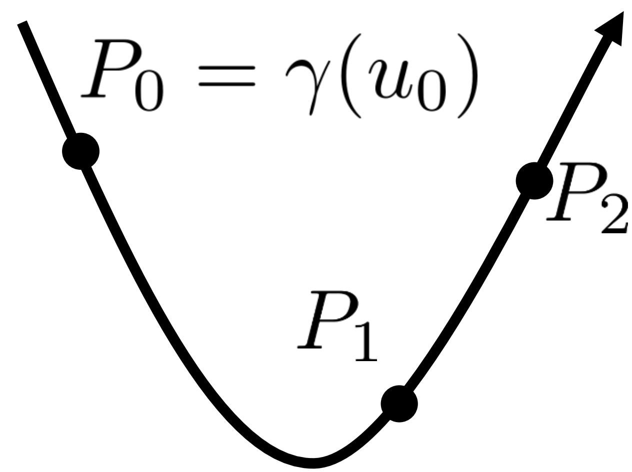
How to pick k? Use k = n

Lagrange Interpolation

Given some points, find polynomial

n points → degree (n-1)

- 2: linear interpolation
- 3: quadratic interp.



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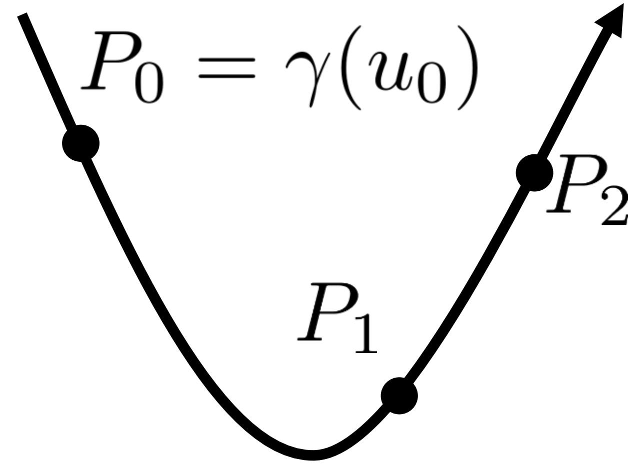

Curves are C^{n-2} smooth

Lagrange Interpolation

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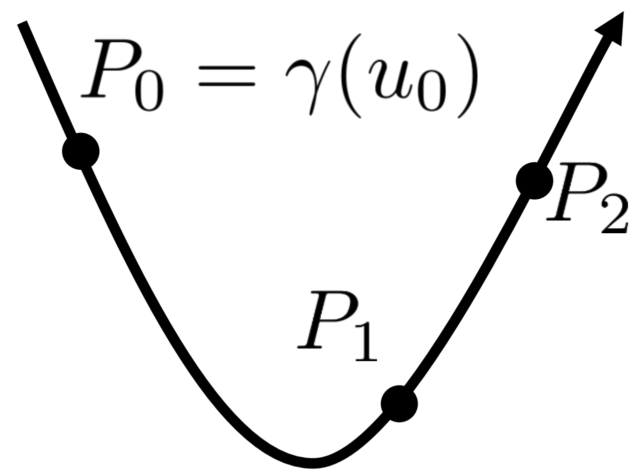
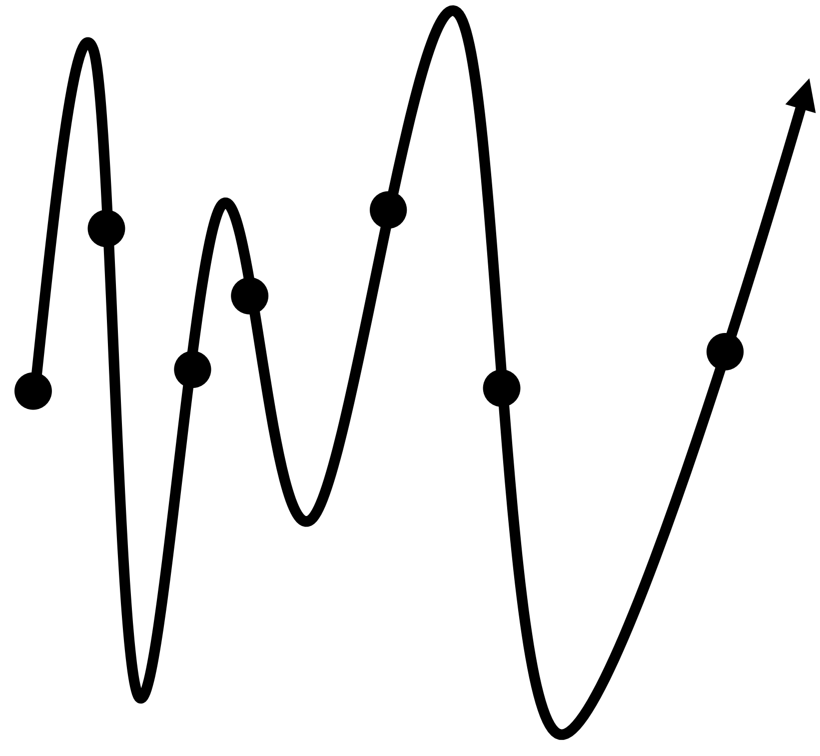


Curves are C^{n-2} smooth

What's the problem?

Lagrange Interpolation

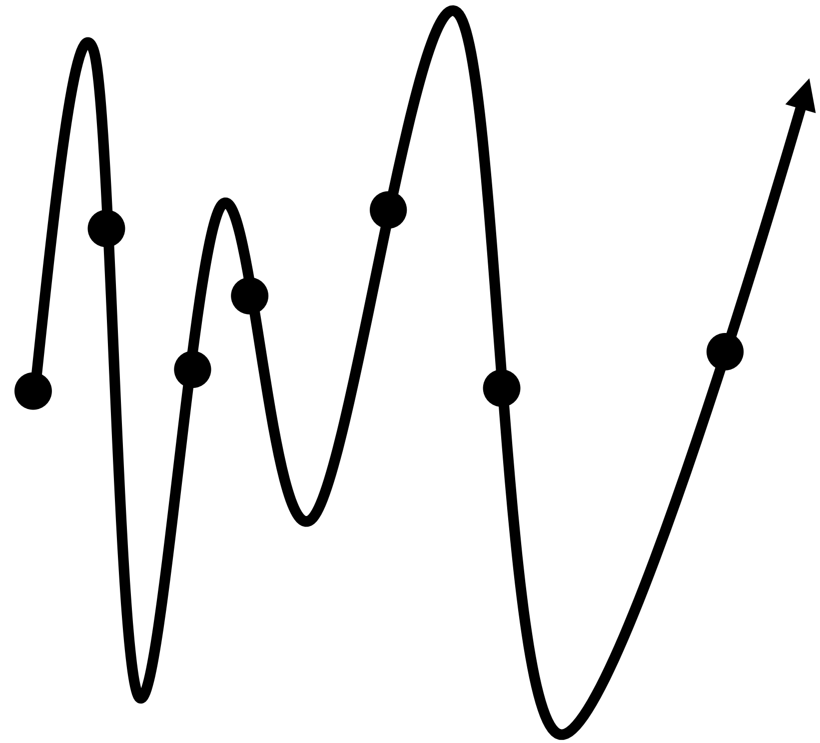
No oscillation control



Lagrange Interpolation

No oscillation control

Worse as degree
becomes larger



Lagrange Interpolation

No oscillation control

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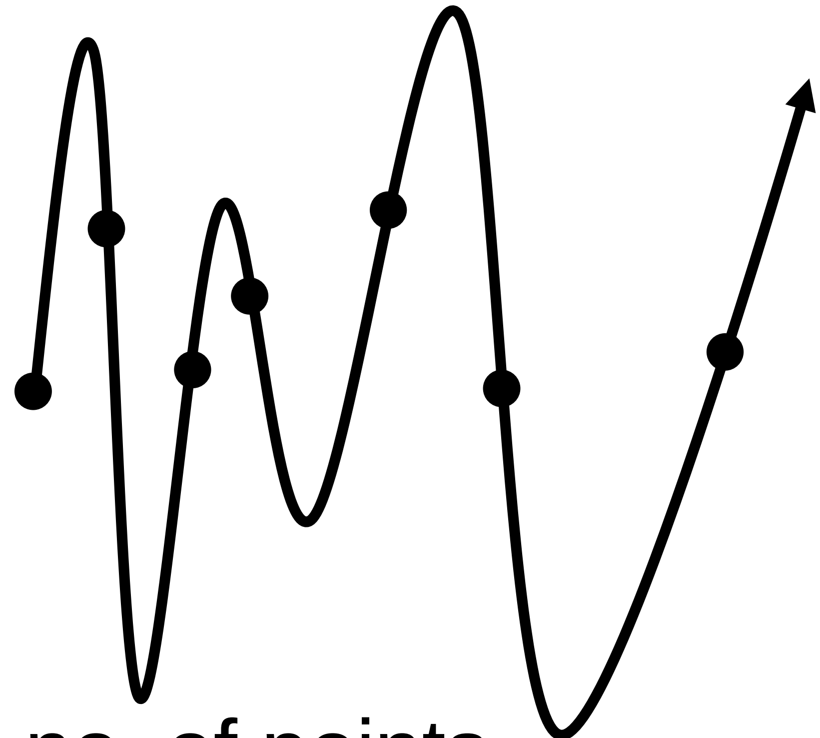
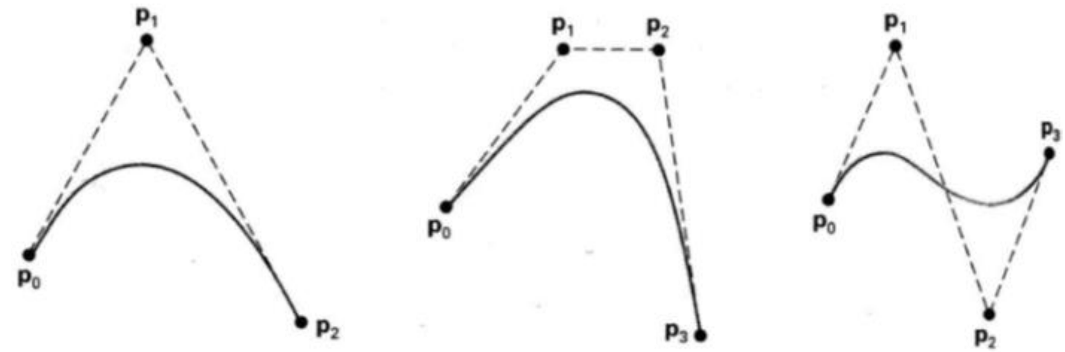


Lagrange interpolation
not practical for large no. of points

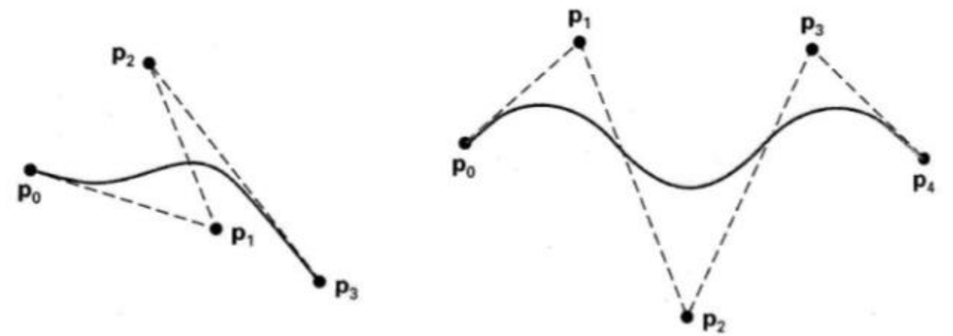
Introducing Splines

Bézier Curves

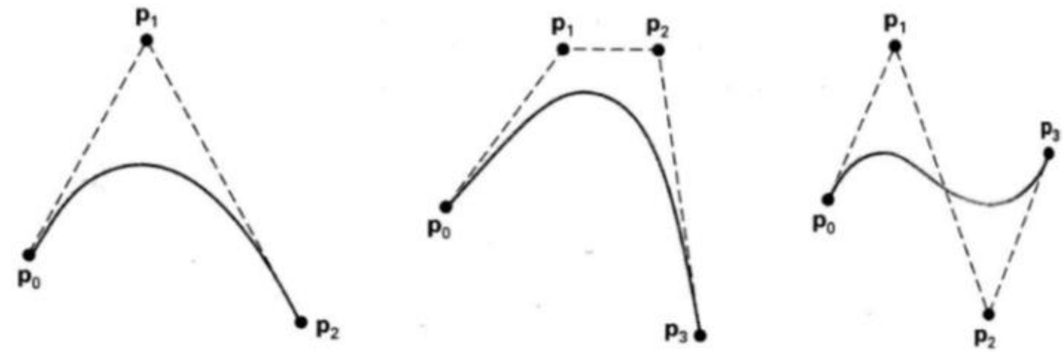


Spline building block

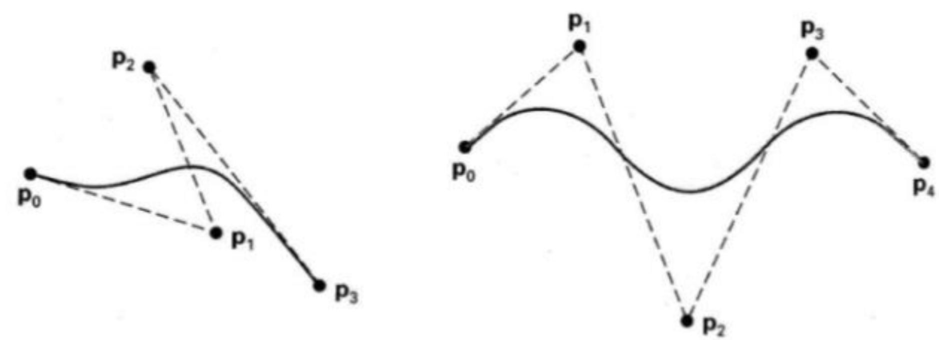
Polynomial



Bézier Curves



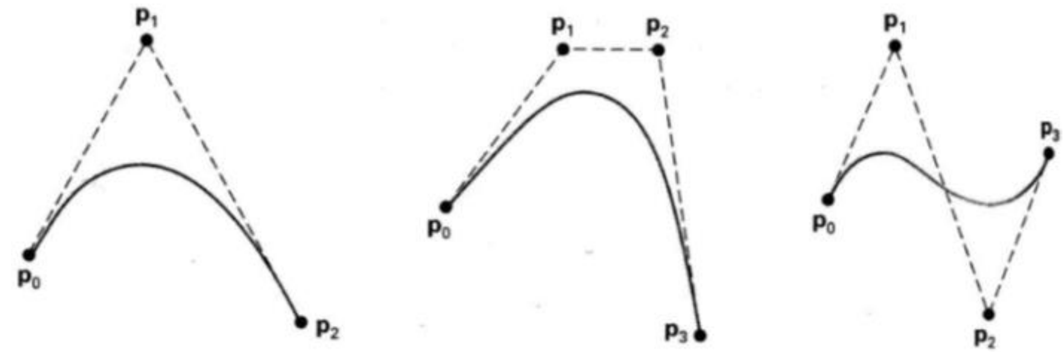
Spline building block



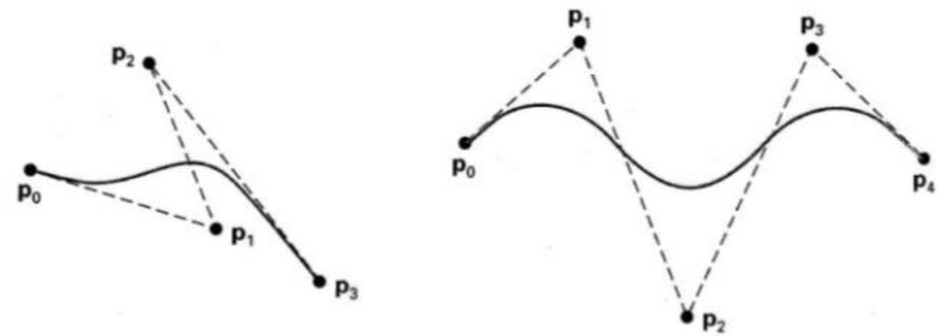
Polynomial

Variation-diminishing: curve lies in convex hull of points

Bézier Curves



Spline building block



Polynomial

Variation-diminishing: curve lies in convex hull of points

Cost: only interpolates endpoints

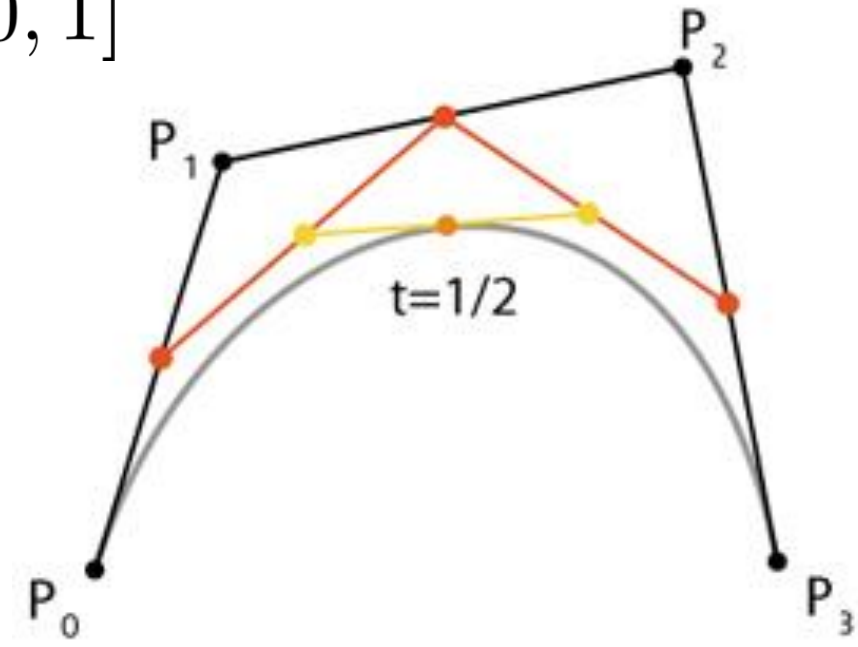
de Casteljau's Algorithm

Given:

- sequence of **control points** P_i
- **single** value of $t \in [0, 1]$

Computes:

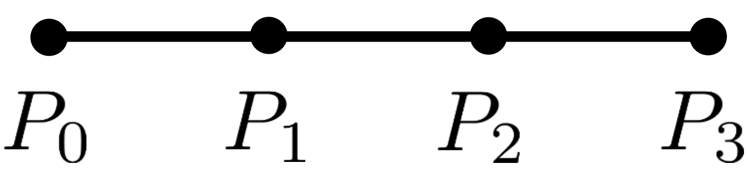
- location of $\gamma(t)$

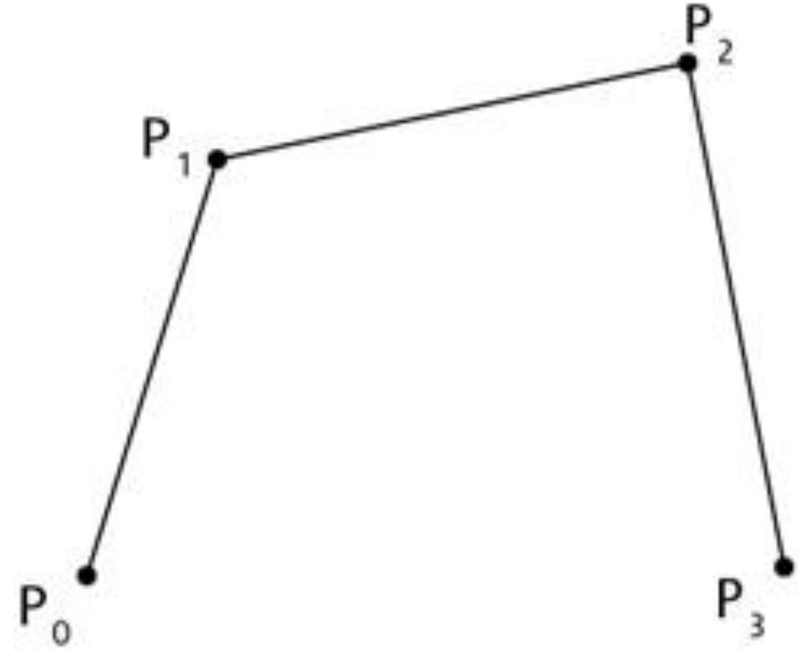


de Casteljau's Algorithm

Main idea: recursive linear interpolation

Start with four points – **control polygon**



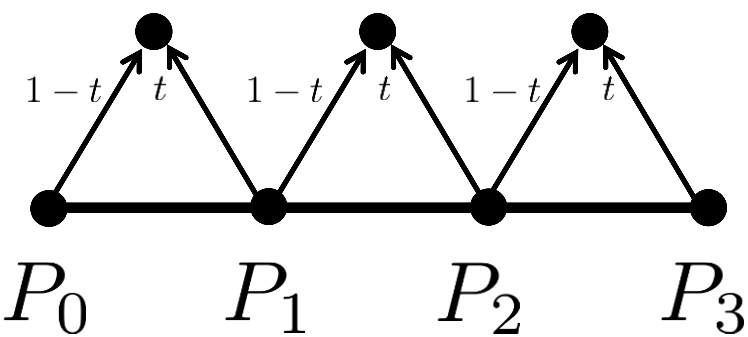


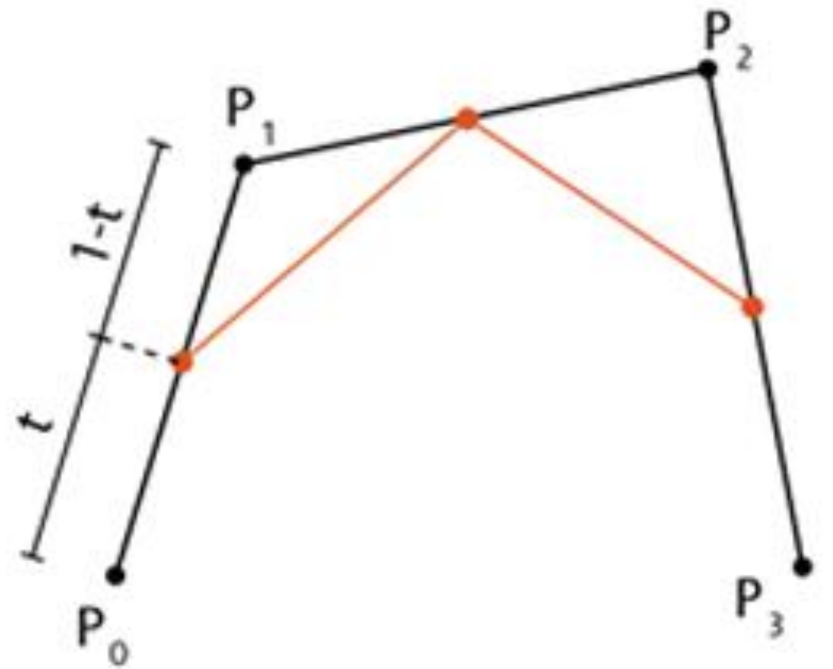
de Casteljau's Algorithm

Main idea: recursive linear interpolation

Start with four points – **control polygon**

Clip corners





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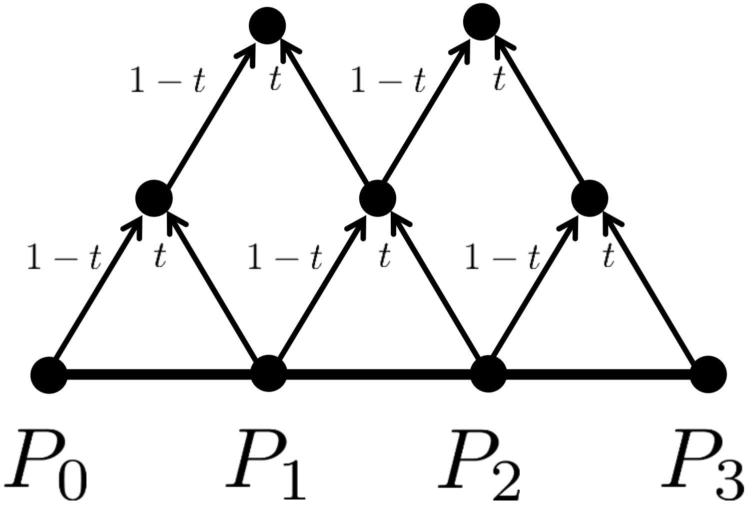
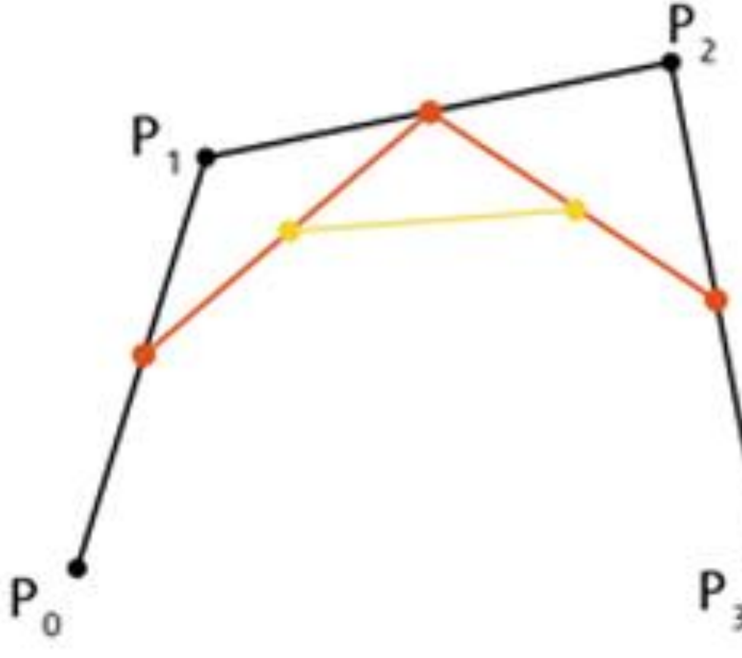
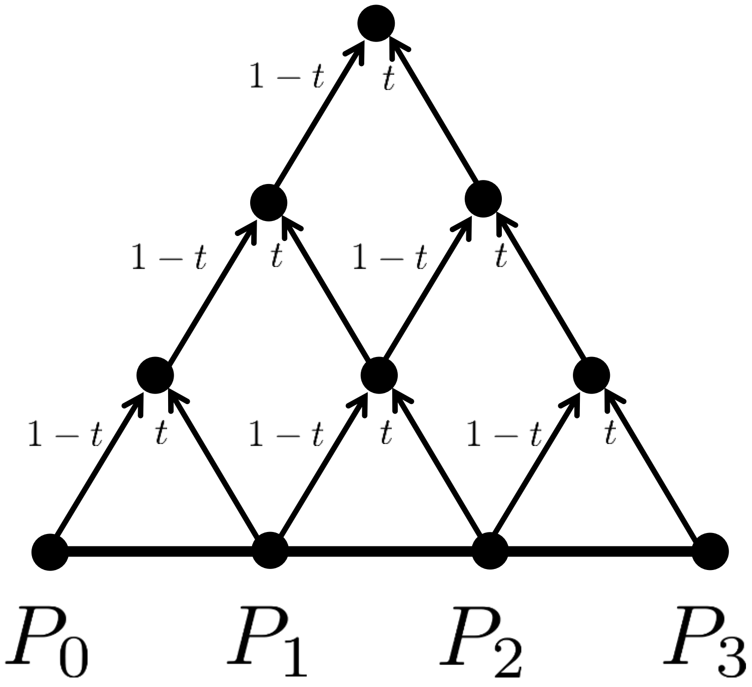
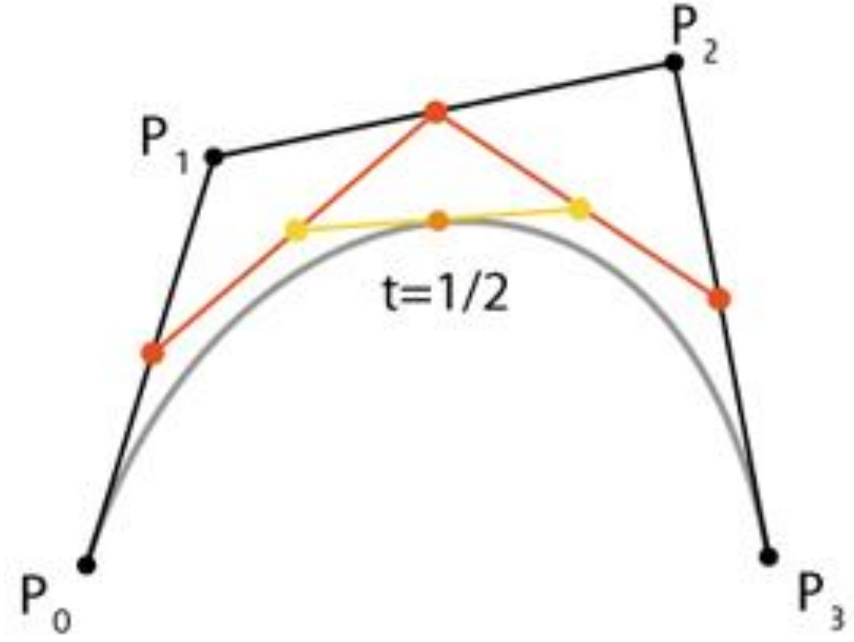




de Casteljau's Algorithm

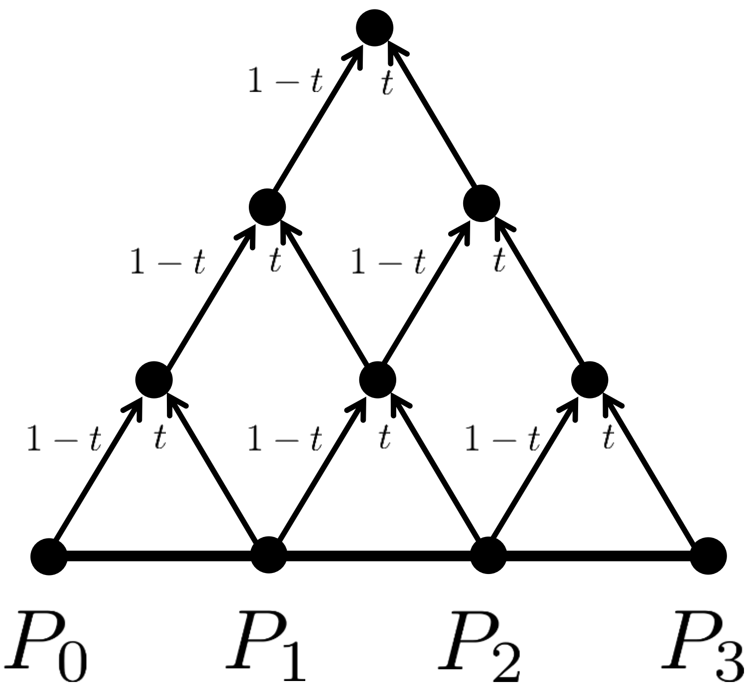
Four control points → cubic Bézier curve

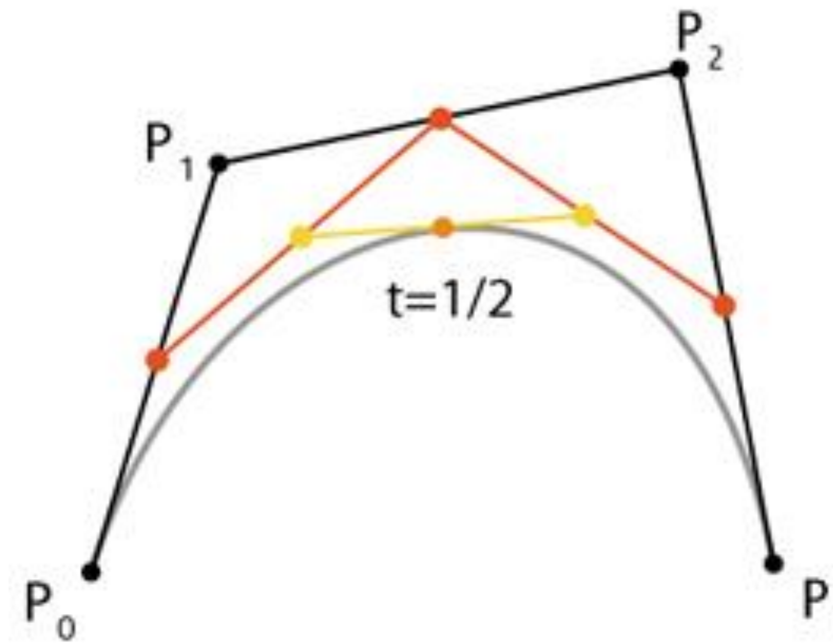




de Casteljau's Algorithm

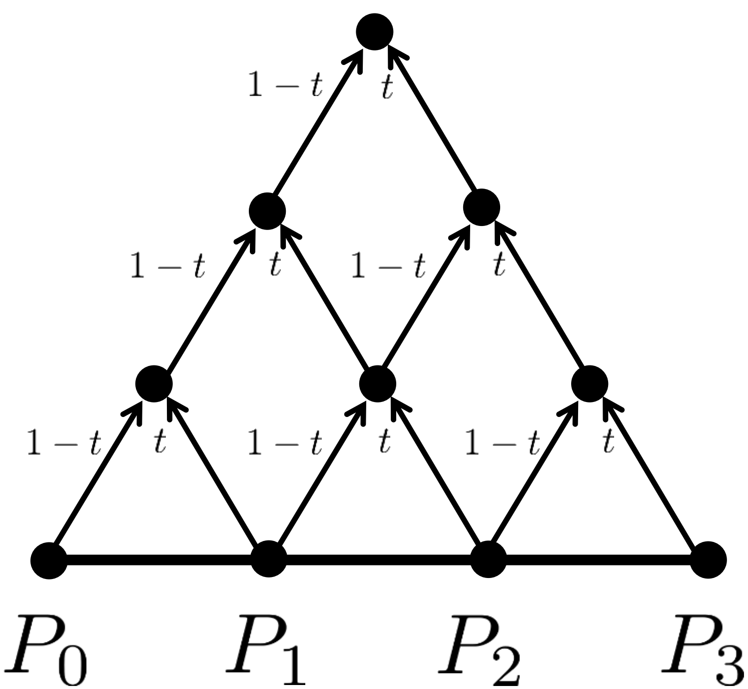
More control points → smoother curve
(more pyramid levels)





de Casteljau's Algorithm

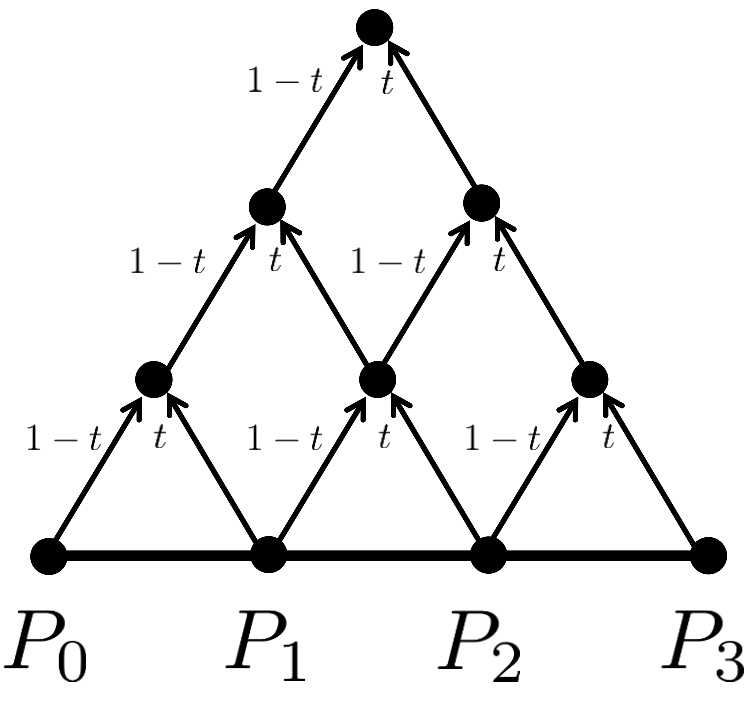
Time complexity?



de Casteljau's Algorithm

Time complexity?

O(n^2) for each evaluation



de Casteljau's Algorithm

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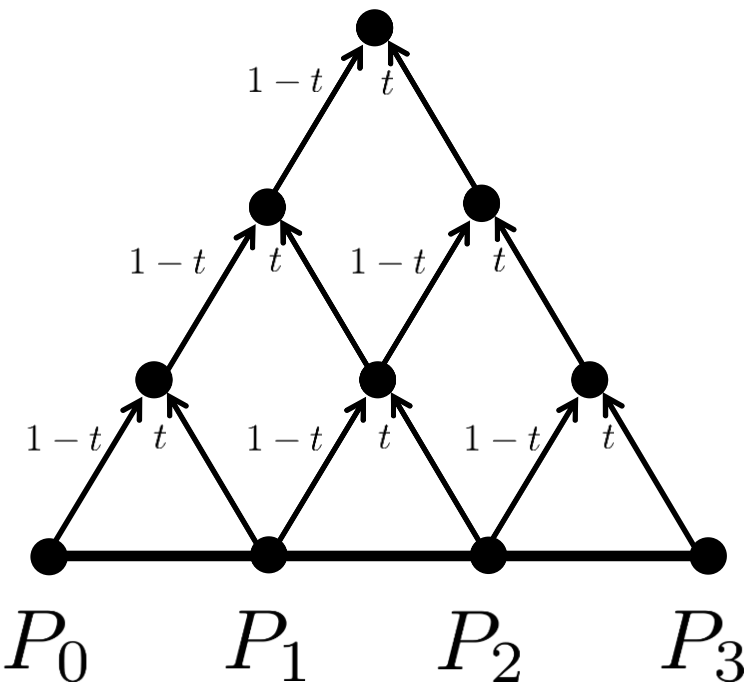
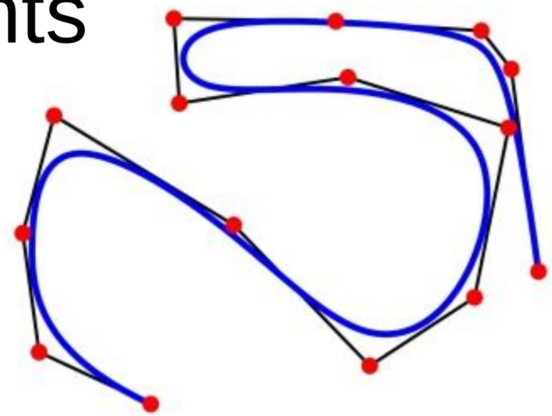


O(n^2) for each evaluation

Also, for long curve, may not want global influence of control points

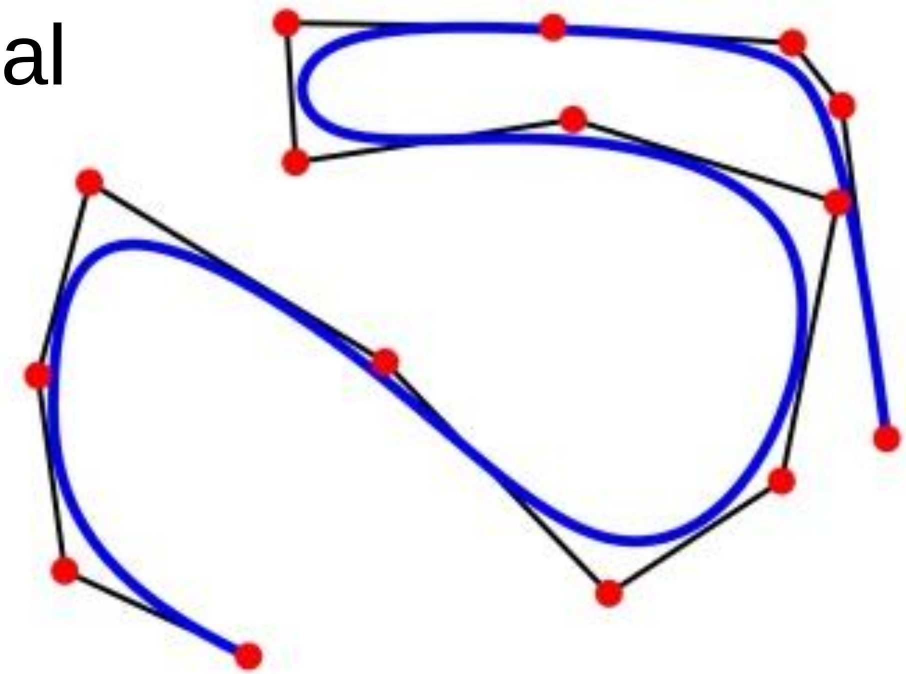


B-Splines ("Basis Splines")

Piecewise polynomial

- (cubic common)

Used in Illustrator, Inkscape, etc



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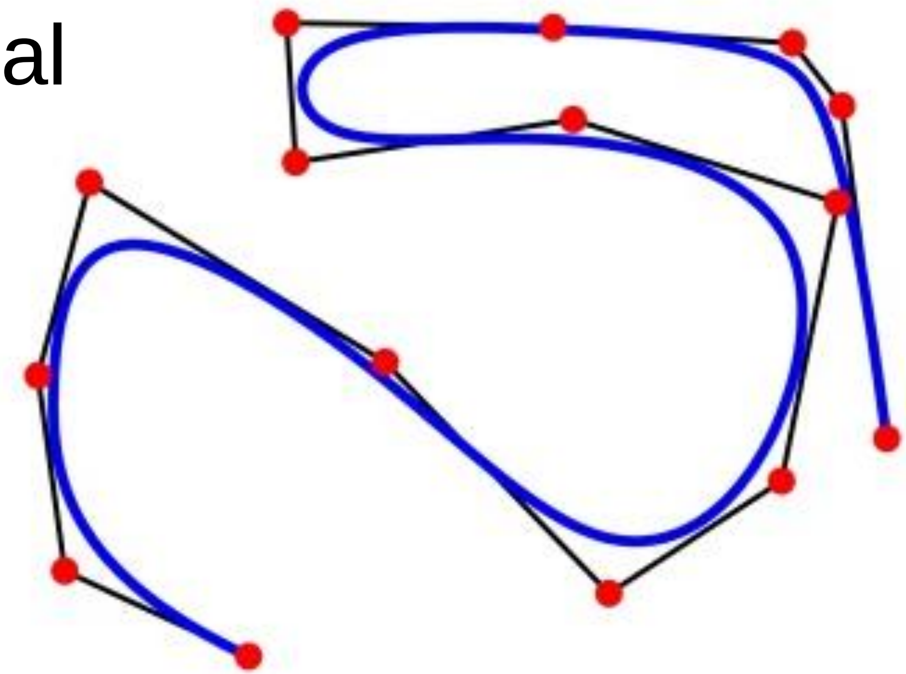


Used in Illustrator,
Inkscape, etc

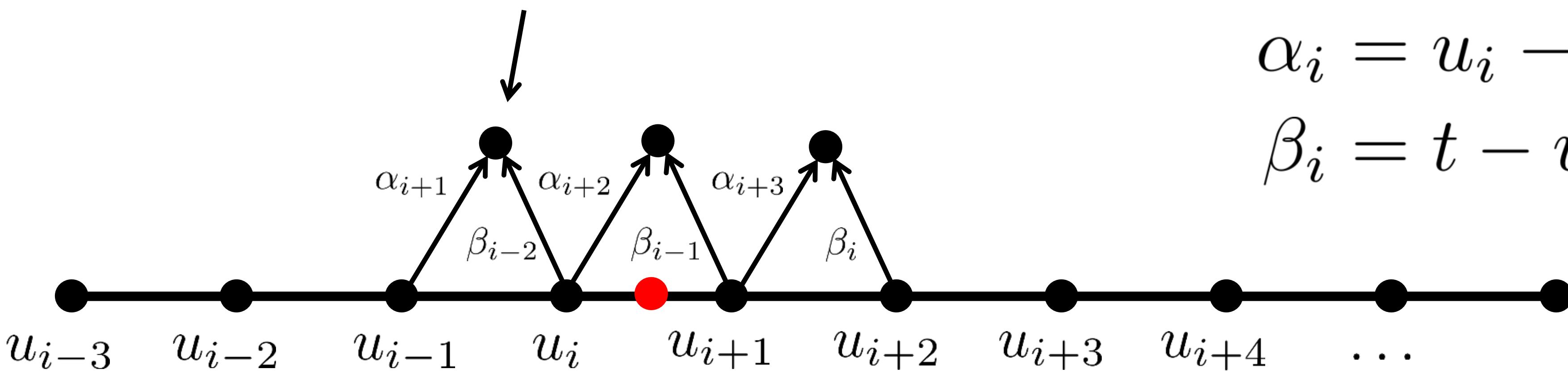
Arbitrary number of control points

- only first and last interpolated

de Boor’s Algorithm

Pyramid algorithm, like de Casteljau

$$\alpha_i = u_i - t$$
$$\beta_i = t - u_i$$



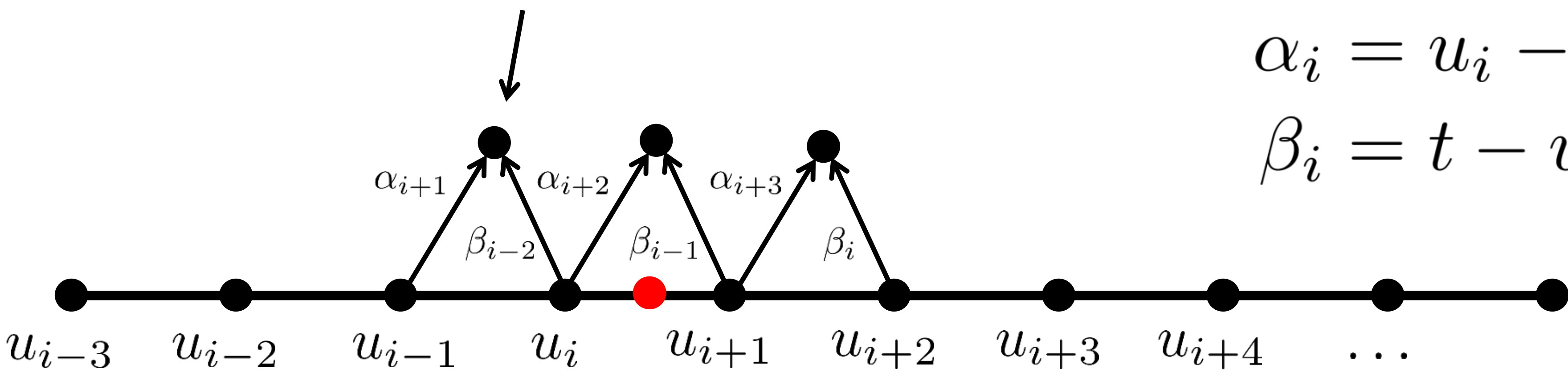
de Boor's Algorithm

Pyramid algorithm, like de Casteljau

$$\frac{\alpha_{i+1}}{\beta_{i-2}+\alpha_{i+1}} P_{i-1} + \frac{\beta_{i-2}}{\beta_{i-2}+\alpha_{i+1}} P_i$$

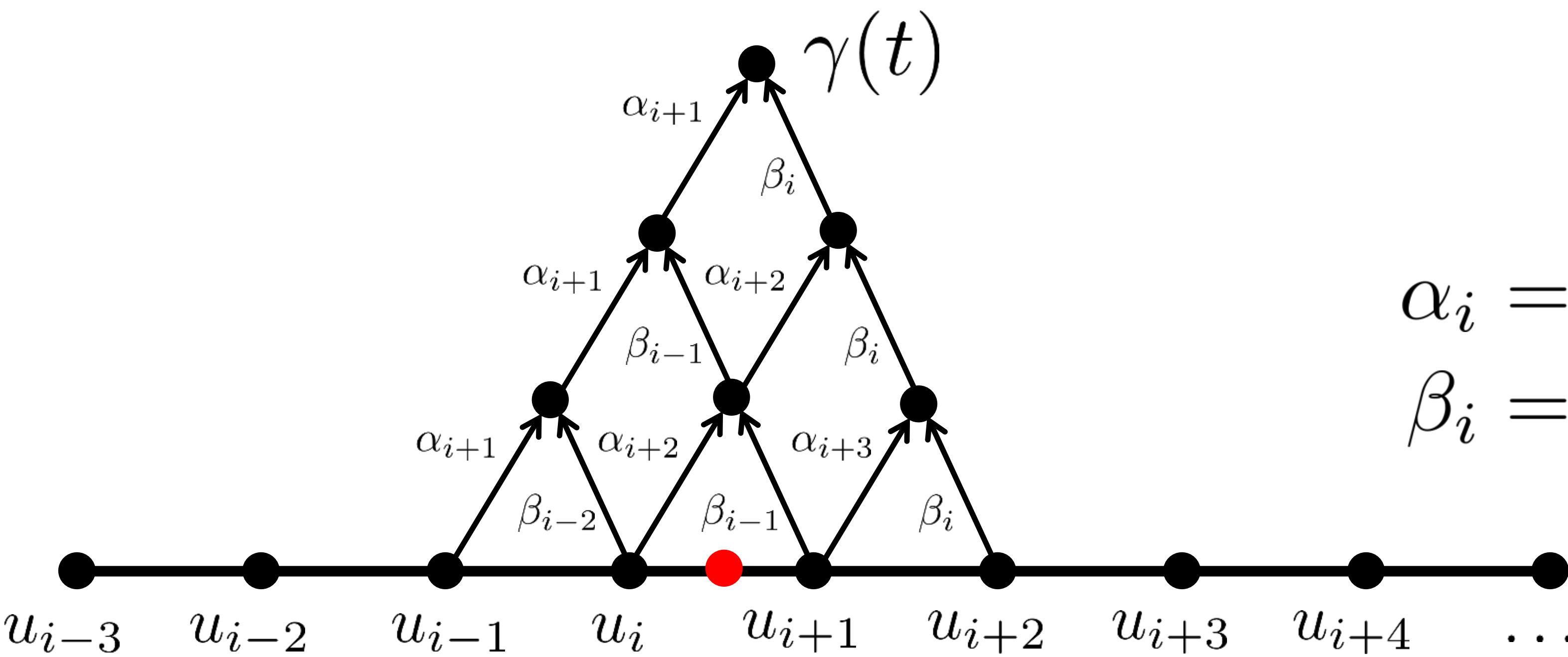
$$\alpha_i = u_i - t$$

$$\beta_i = t - u_i$$



de Boor's Algorithm

Pyramid algorithm, like de Casteljau

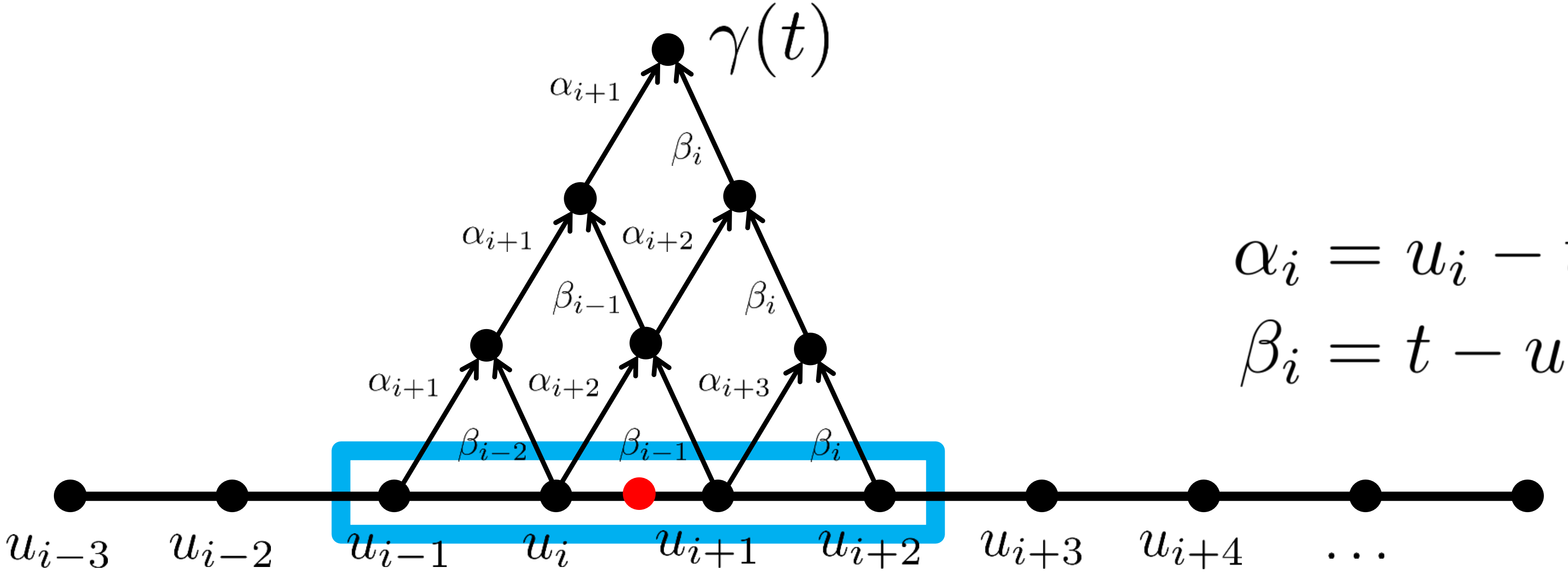


$$\alpha_i = u_i - t$$
$$\beta_i = t - u_i$$

de Boor's Algorithm

Pyramid algorithm, like de Casteljau

Final answer depends on **four** control pts



$$\alpha_i = u_i - t$$

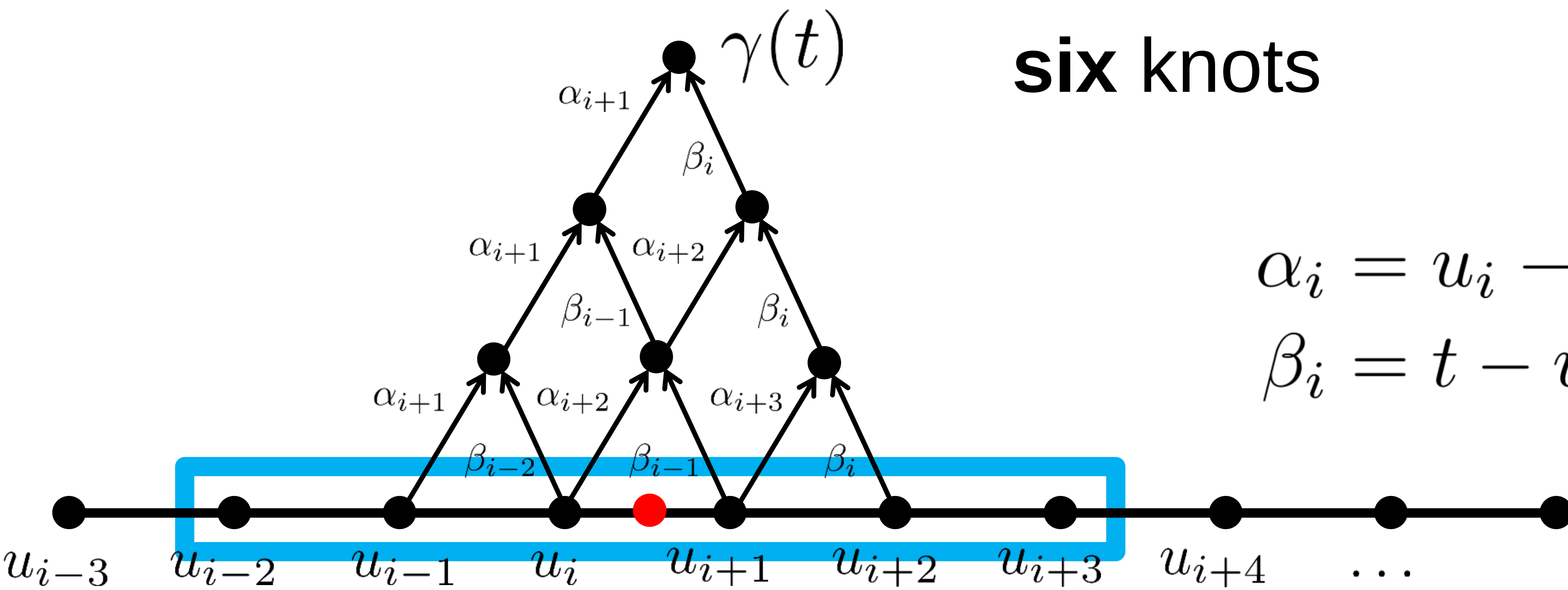
$$\beta_i = t - u_i$$

de Boor's Algorithm

Pyramid algorithm, like de Casteljau

Final answer depends on **four** control pts

six knots

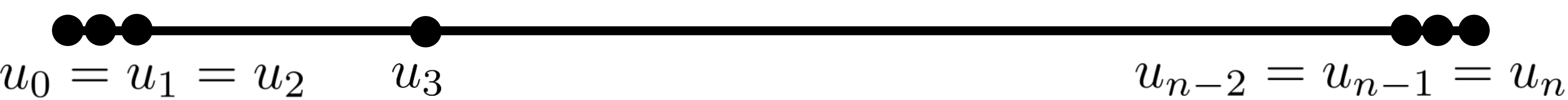


$$\alpha_i = u_i - t$$

$$\beta_i = t - u_i$$

de Boor's Algorithm

Knots triplicated at boundaries



de Boor's Algorithm

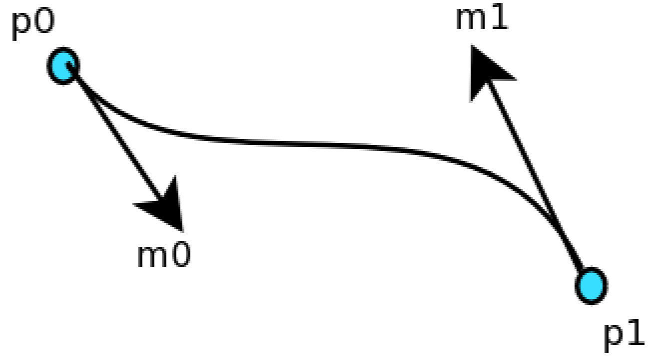
Knots triplicated at boundaries

Higher degree → more pyramid levels
more duplicates at bdry

Other Spline Types

Hermite

- can also specify derivatives at boundary



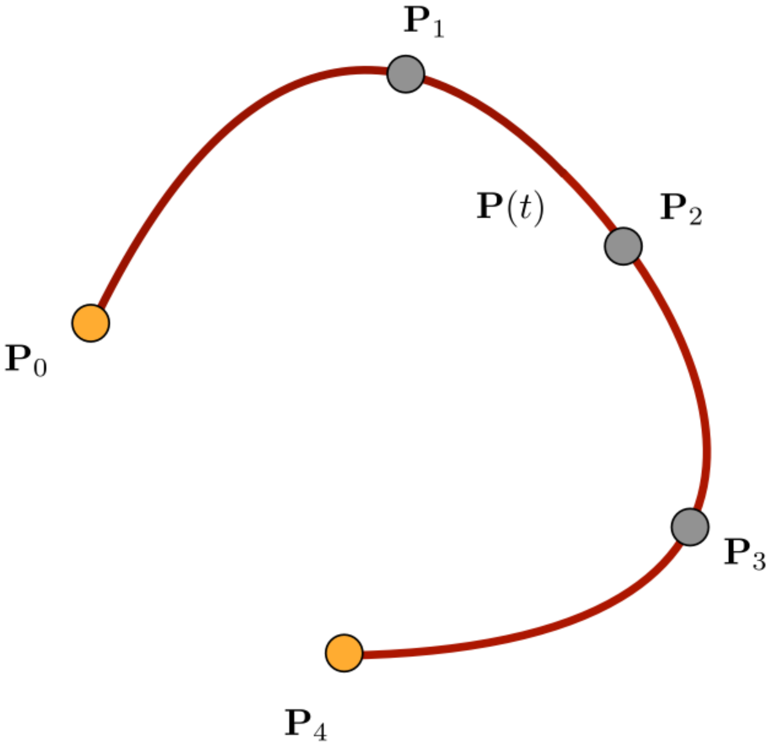
Other Spline Types

Hermite

- can also specify derivatives at boundary

Catmull-Rom

- interpolatory



Spline Keywords

Interpolatory

- spline goes through all control points

Linear

- curve pts linear in **control points**

Degree **n**

- curve pts depend on **n**th power of **t**

Uniform

- knots evenly spaced