Geometry Processing

What is Geometry Processing?

Understanding the math of **3D shape**…

…and applying that math to **discrete** shape

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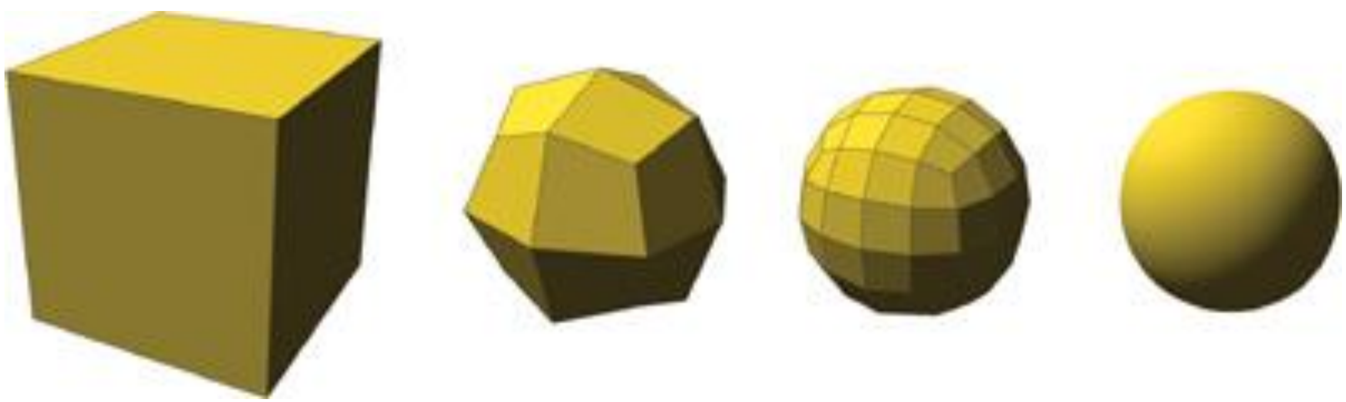
Examples:

- subdivision and decimation



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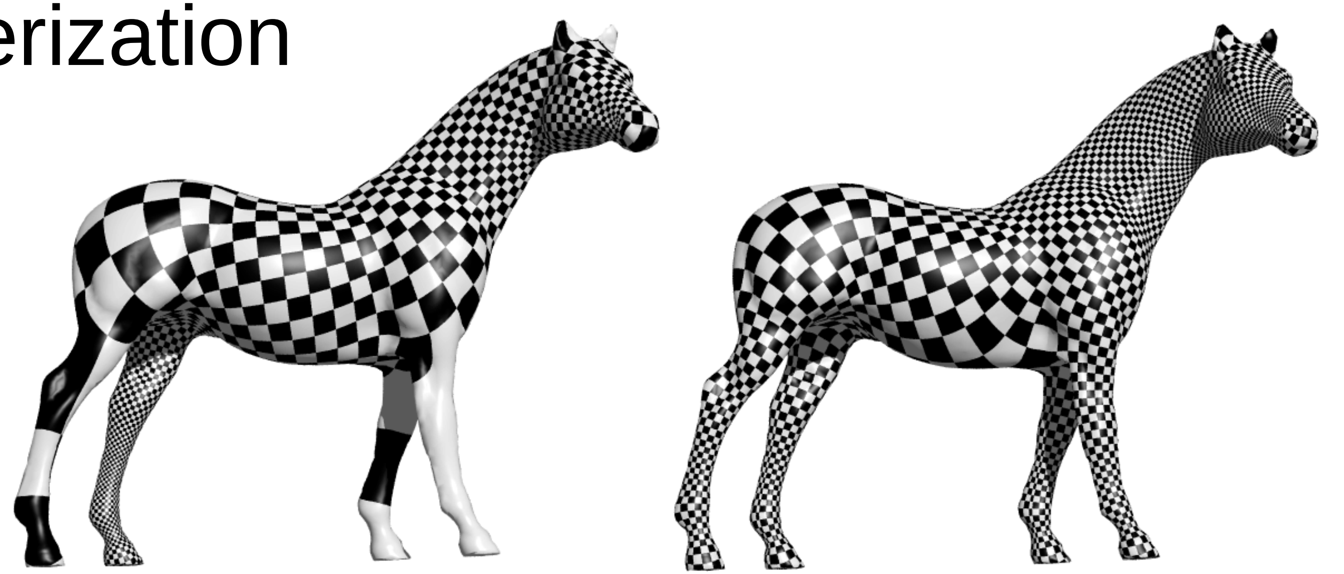

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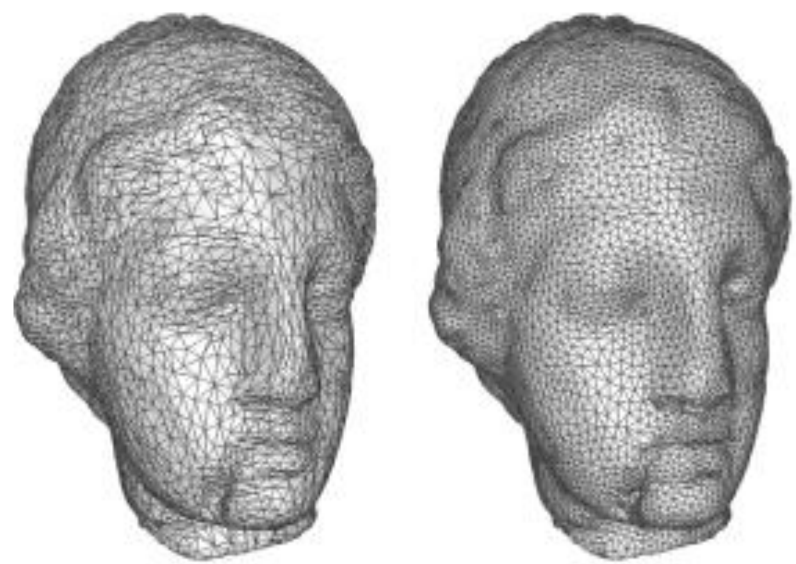

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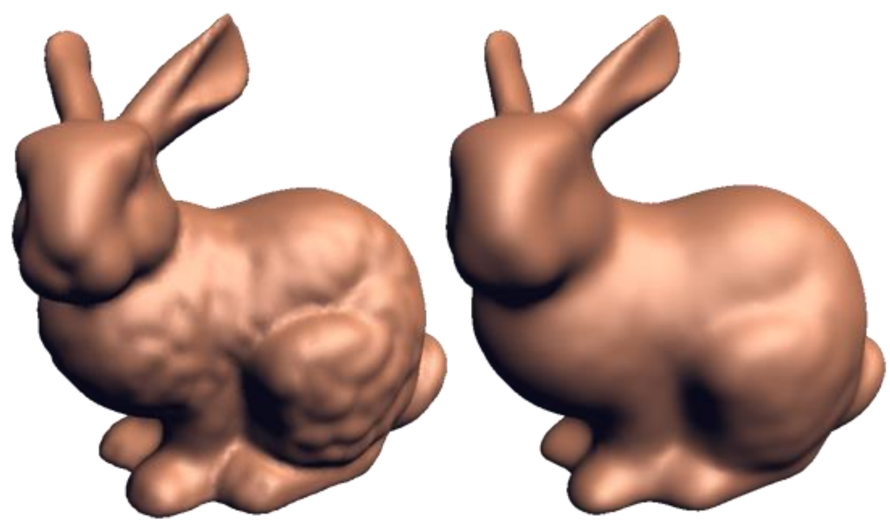

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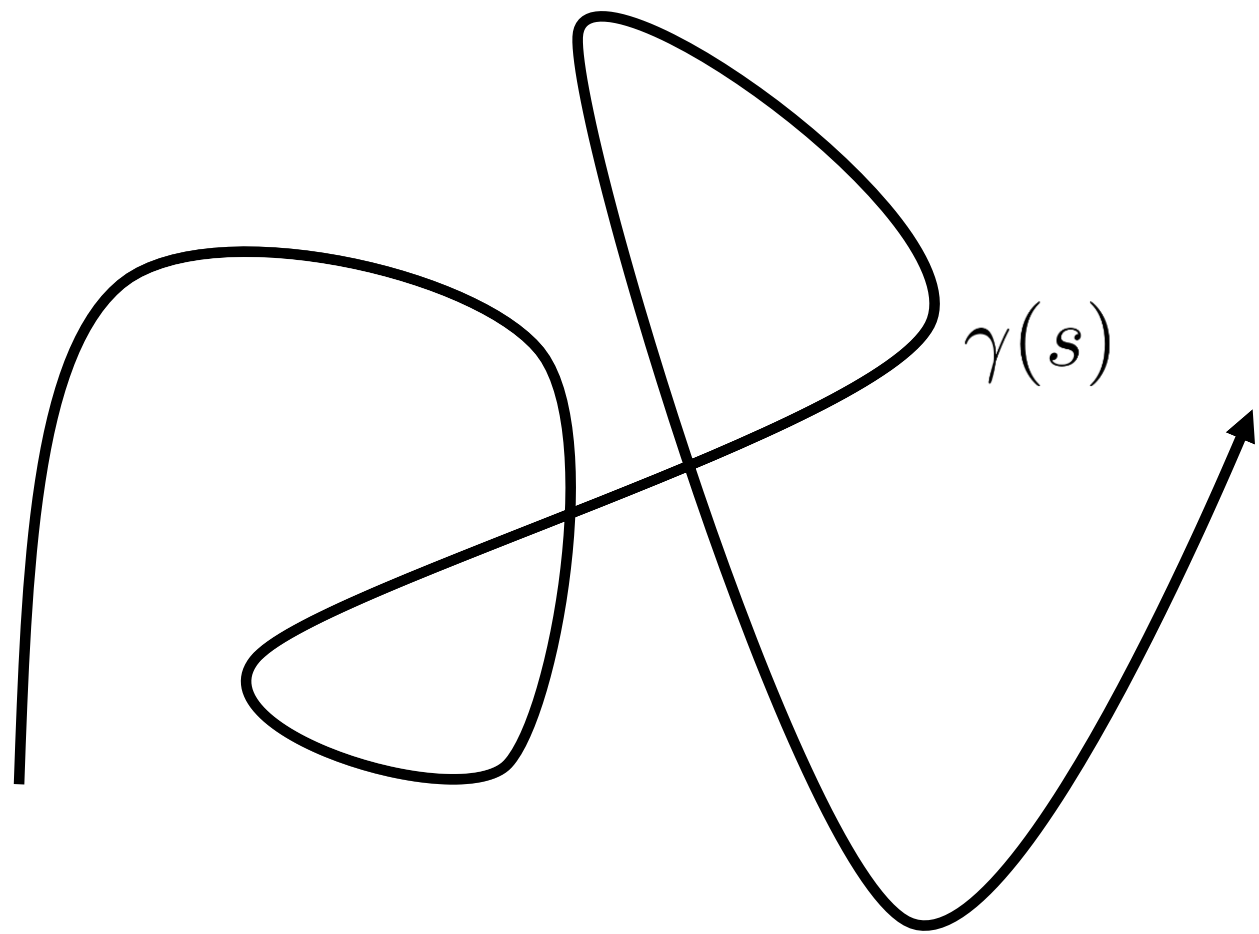
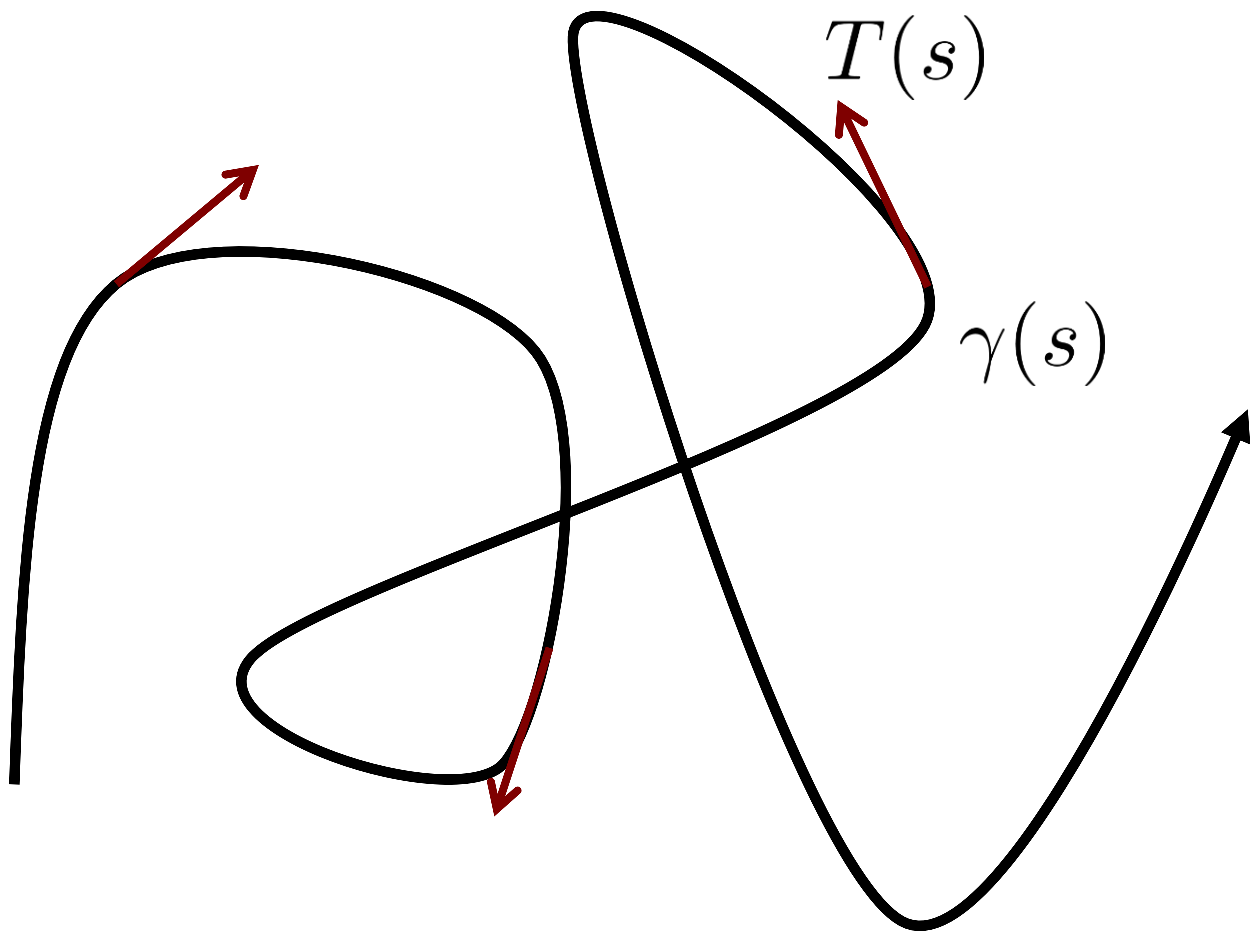
Today: a quick taste of surface geometry

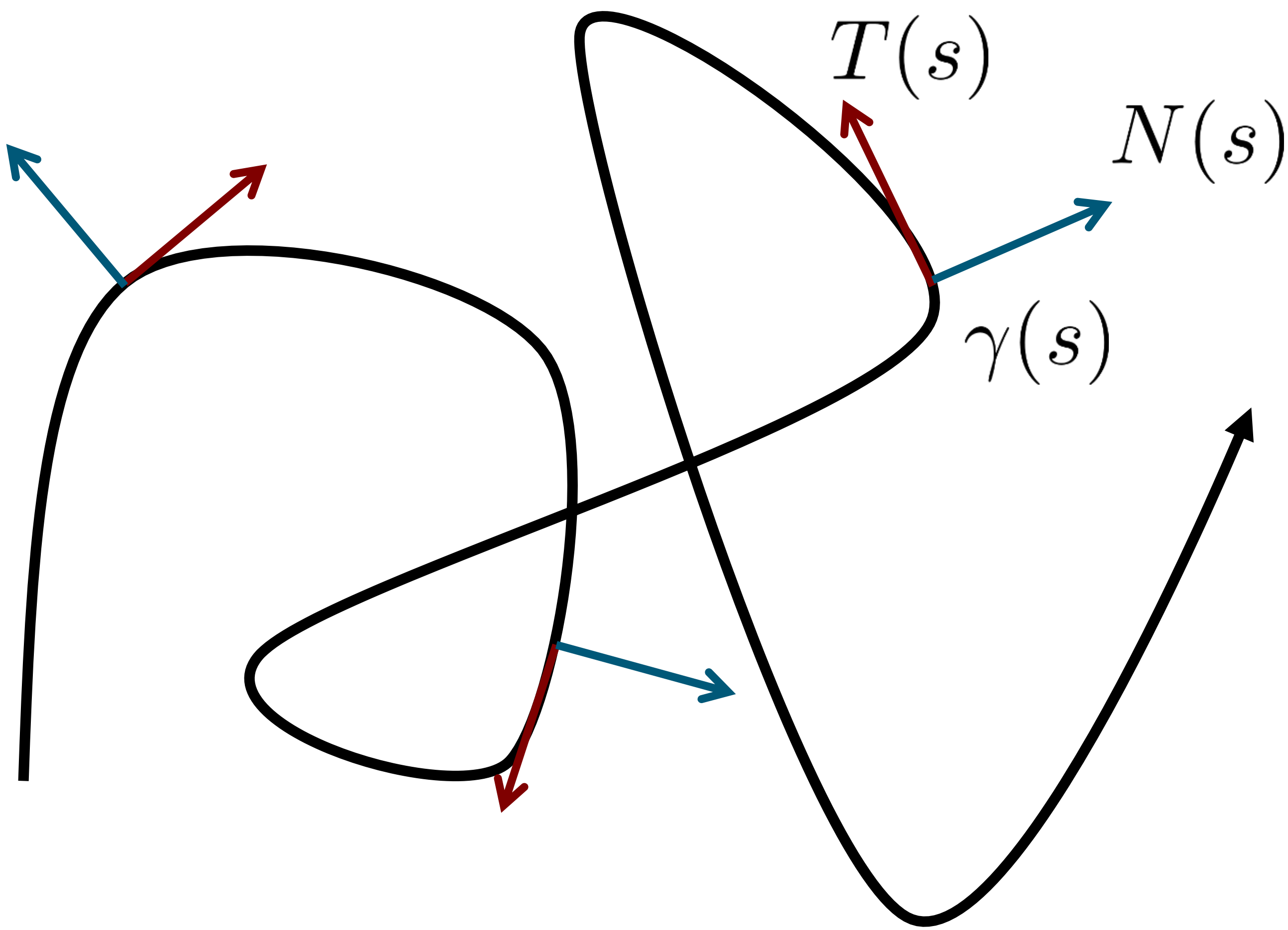
Simple Geometry: Plane Curves



Simple Geometry: Plane Curves

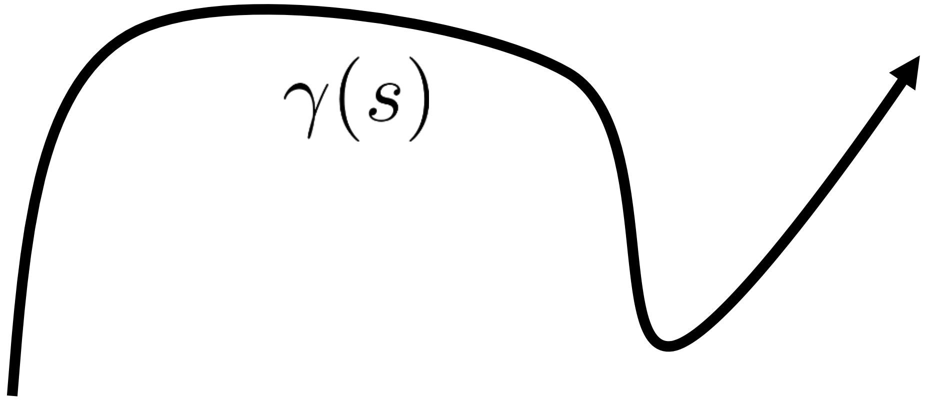


Simple Geometry: Plane Curves



Curvature $k(s)$

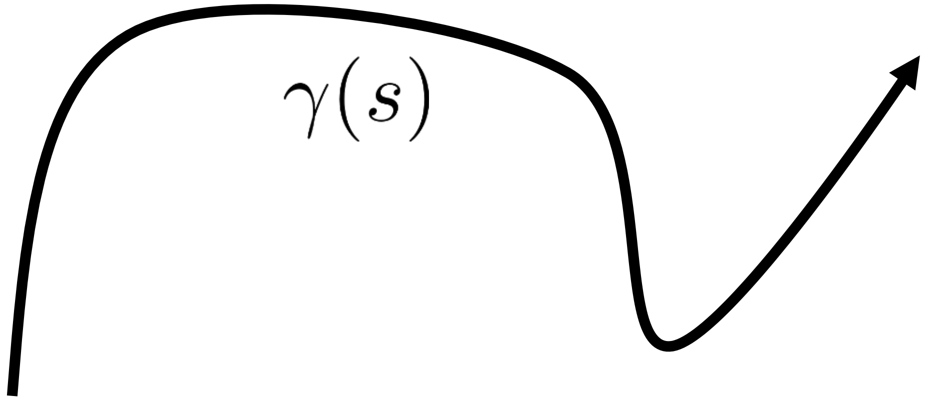
What is it?



Some formula: $\gamma''(s) = -k(s)N(s)$

Curvature $k(s)$

What is it really?



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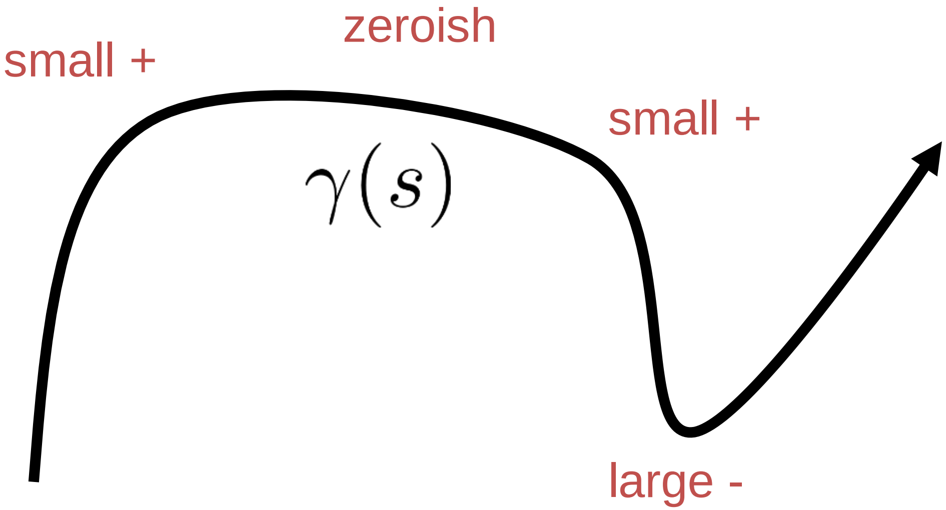

"how quickly the normals turn"

Curvature $k(s)$

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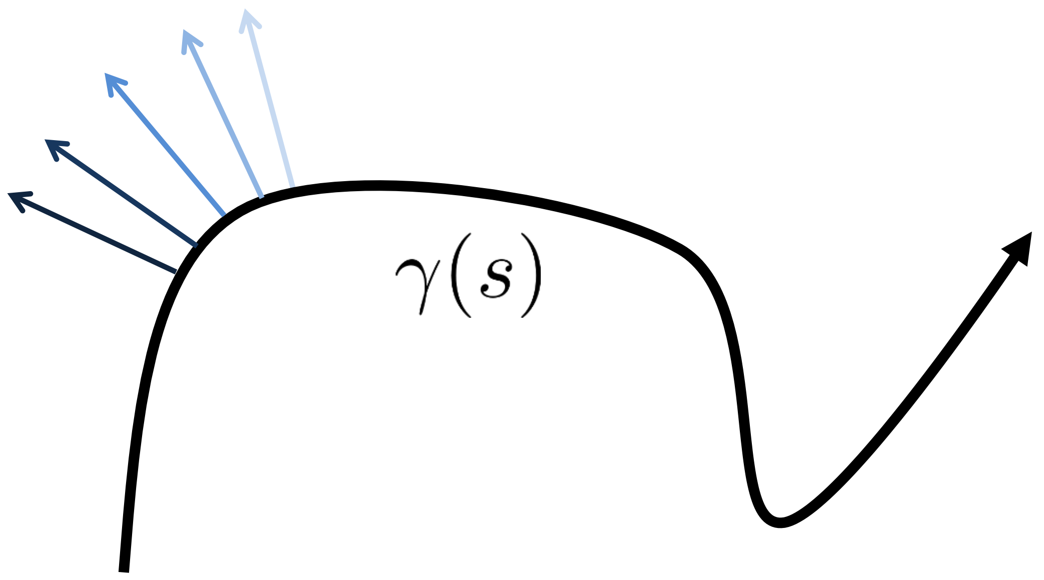

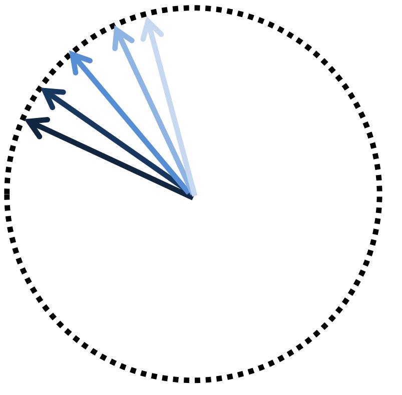

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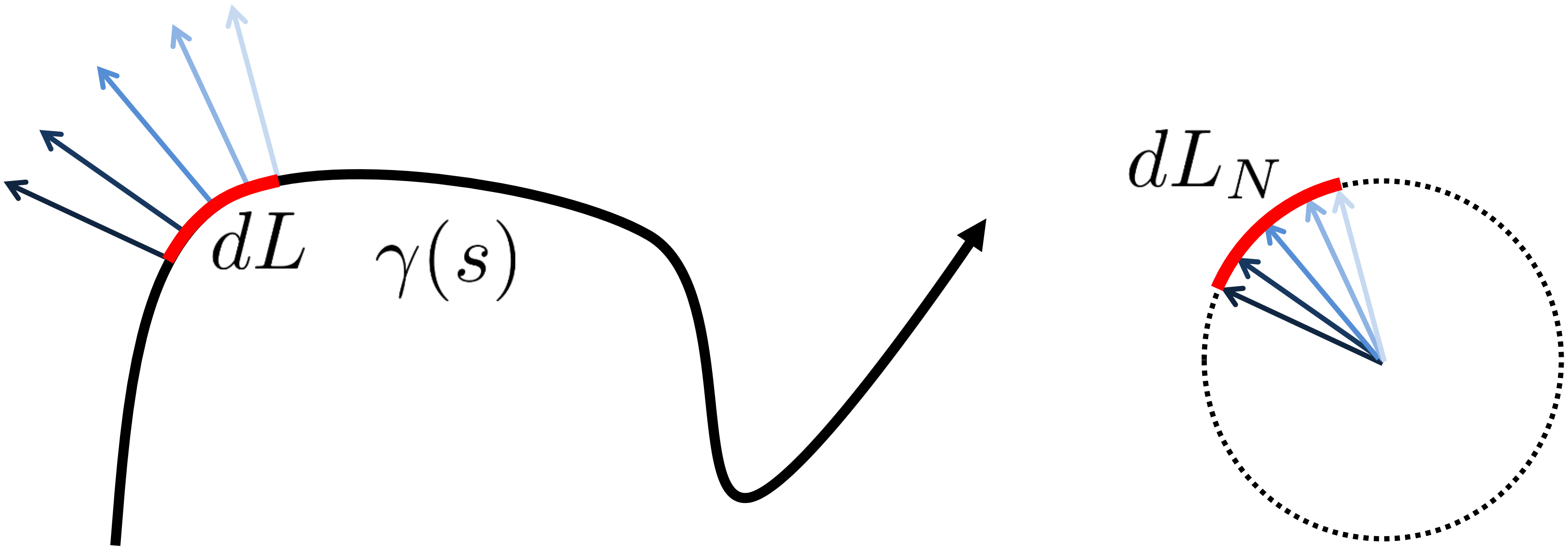
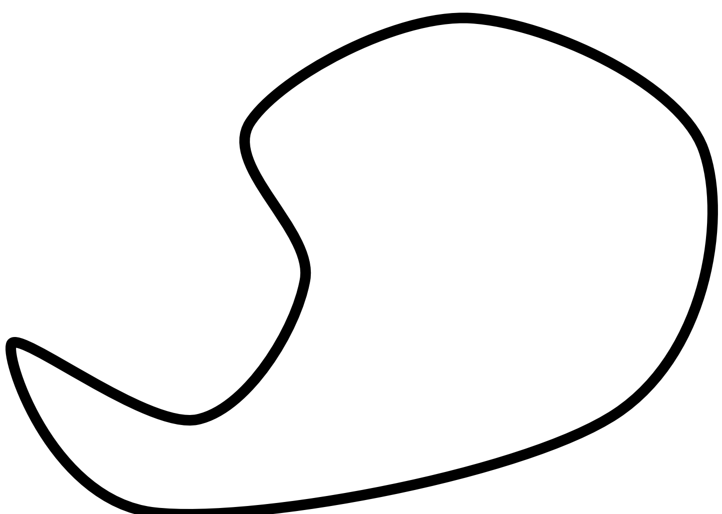


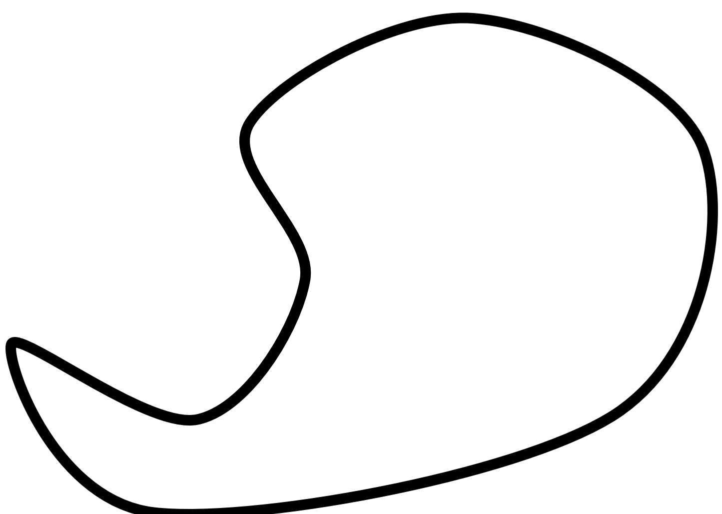
$$k = \frac{dL_N}{dL}$$

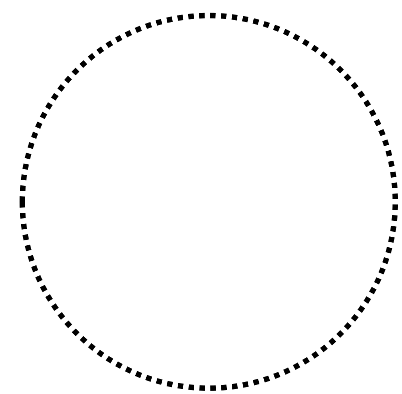
Total Integrated Curvature



$$\int k(s)dL =$$

Total Integrated Curvature

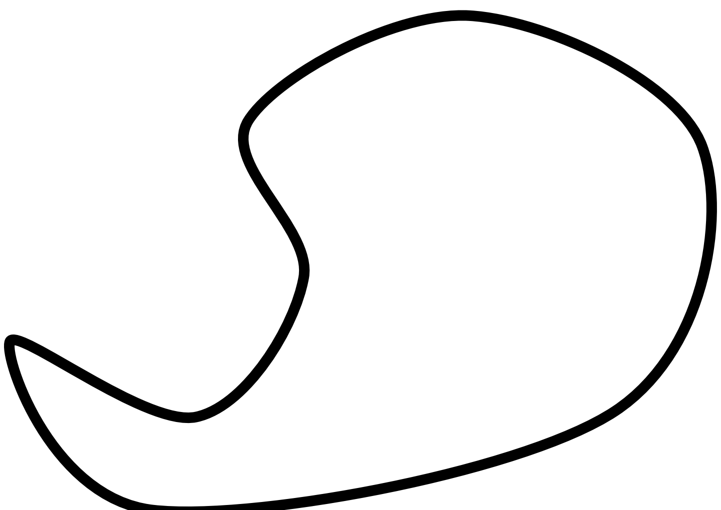


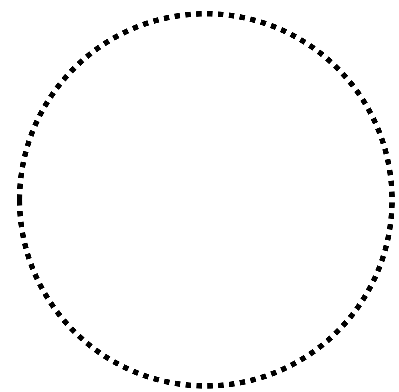


$$\int k(s)dL = \int \frac{dL_N}{dL}dL = 2\pi$$

Total Integrated Curvature

Theorem (Whitney-Graustein): for a closed smooth curve, $\int k(s)dL = 2\pi n$.

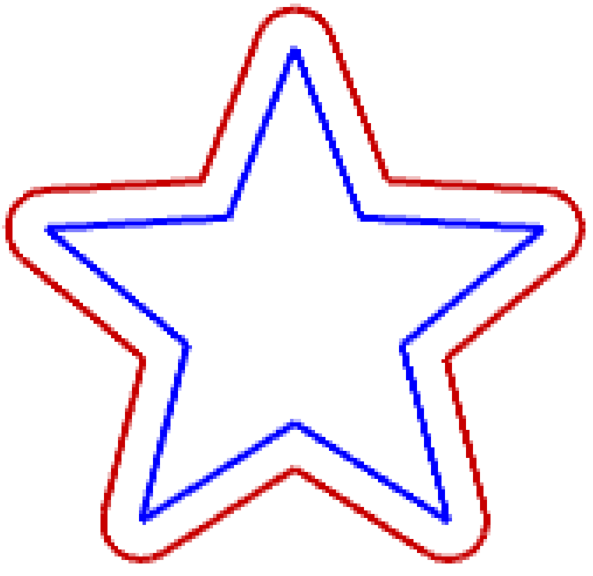




$$\int k(s)dL = \int \frac{dL_N}{dL}dL = 2\pi$$

Inflation Theorem

Offset closed curve along normal direction



Inflation Theorem

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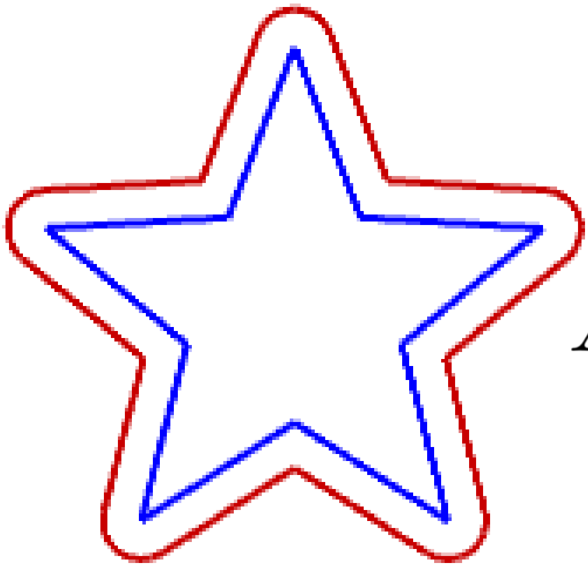
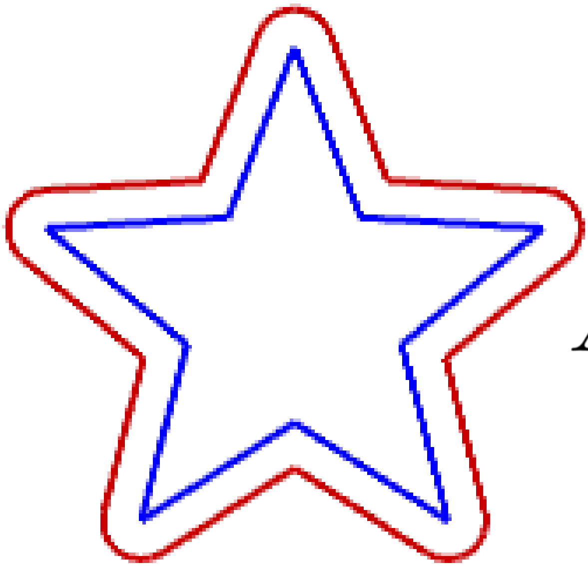


$$A_\epsilon = A$$

Inflation Theorem

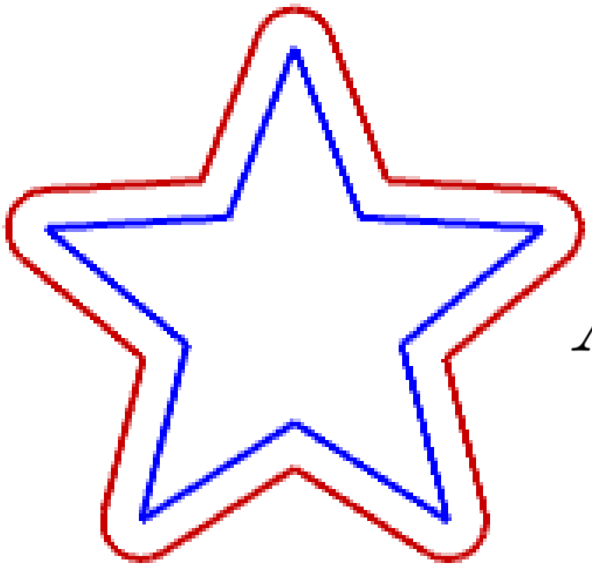
Offset closed curve along normal direction



$$A_\epsilon = A + \epsilon L$$

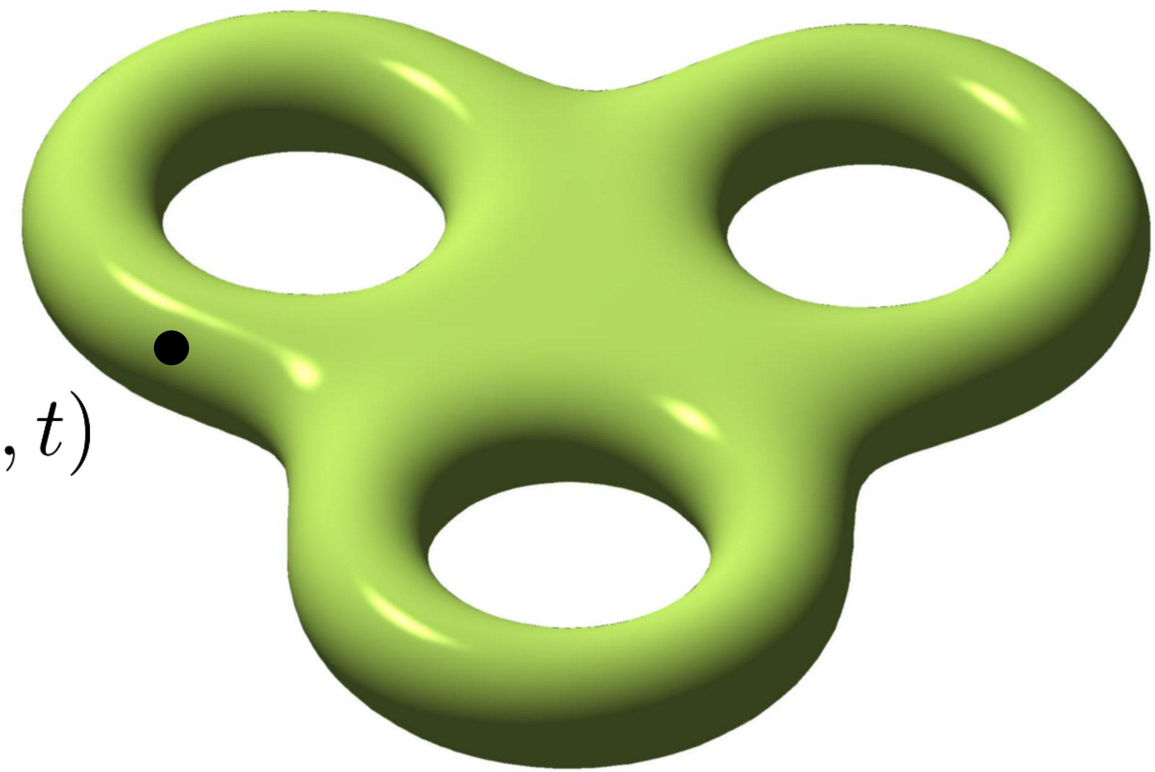
Inflation Theorem

Offset closed curve along normal direction



$$A_\epsilon = A + \epsilon L + \frac{1}{2}\epsilon^2 \int k(s) dL$$

Surfaces in Space

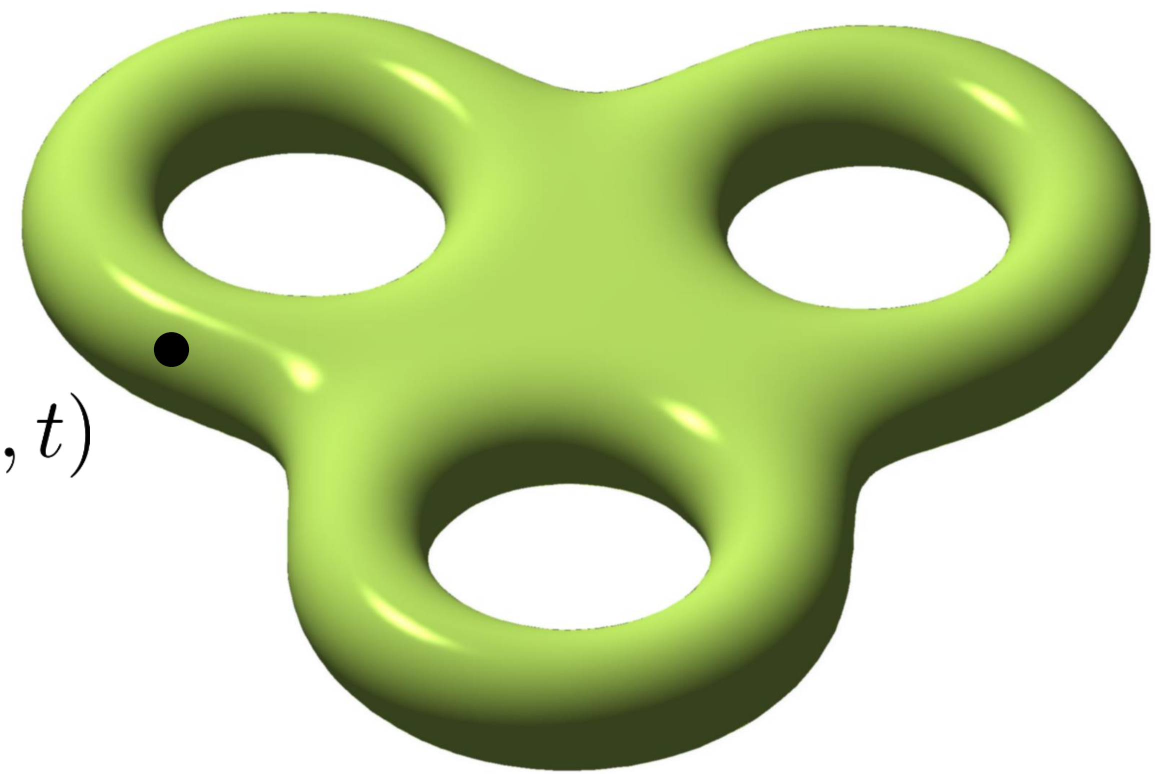


$r(s, t)$

$r : \mathbb{R}^2 \to \mathbb{R}^3$

Surfaces in Space

What is curvature now?



$r(s,t)$

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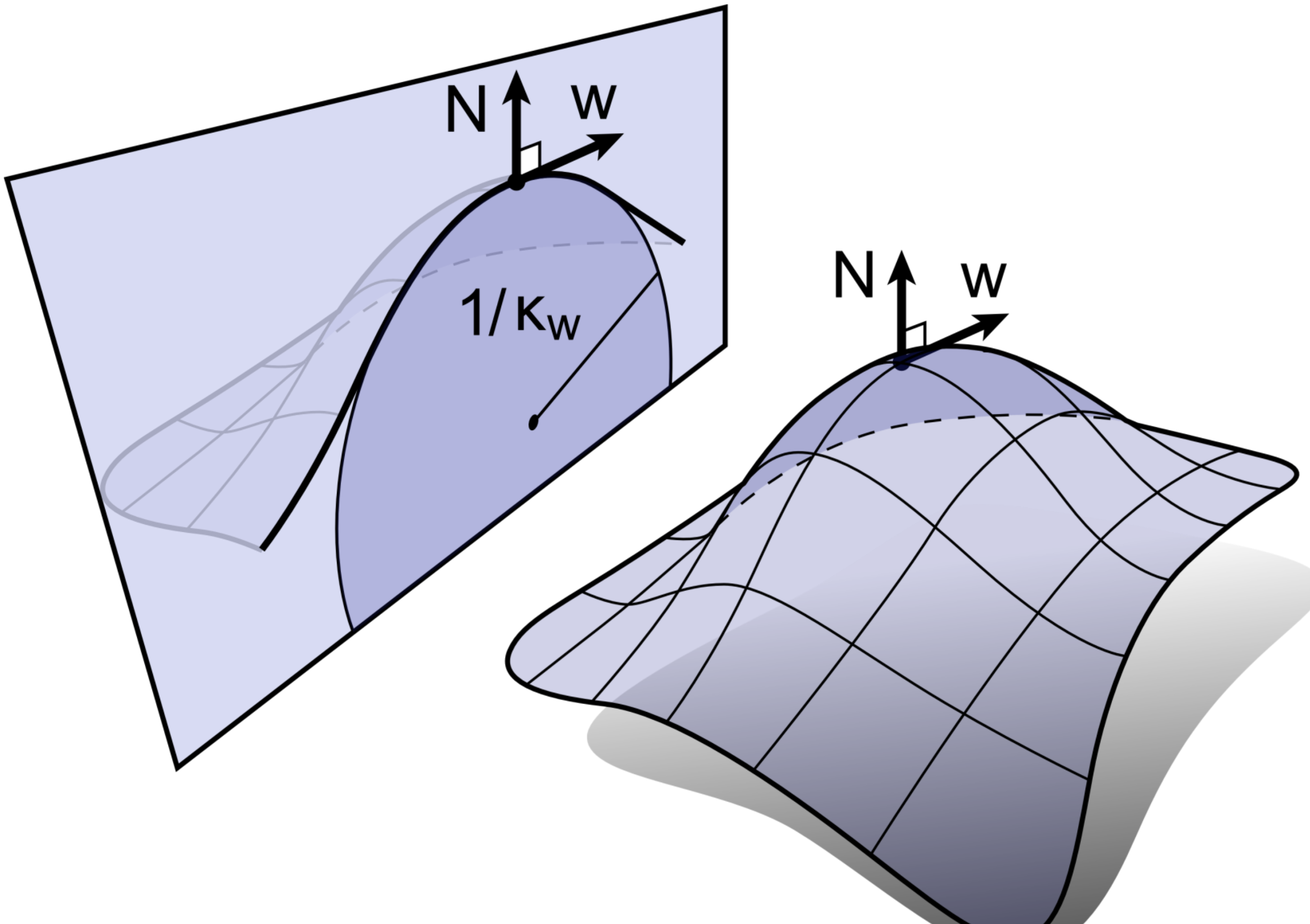
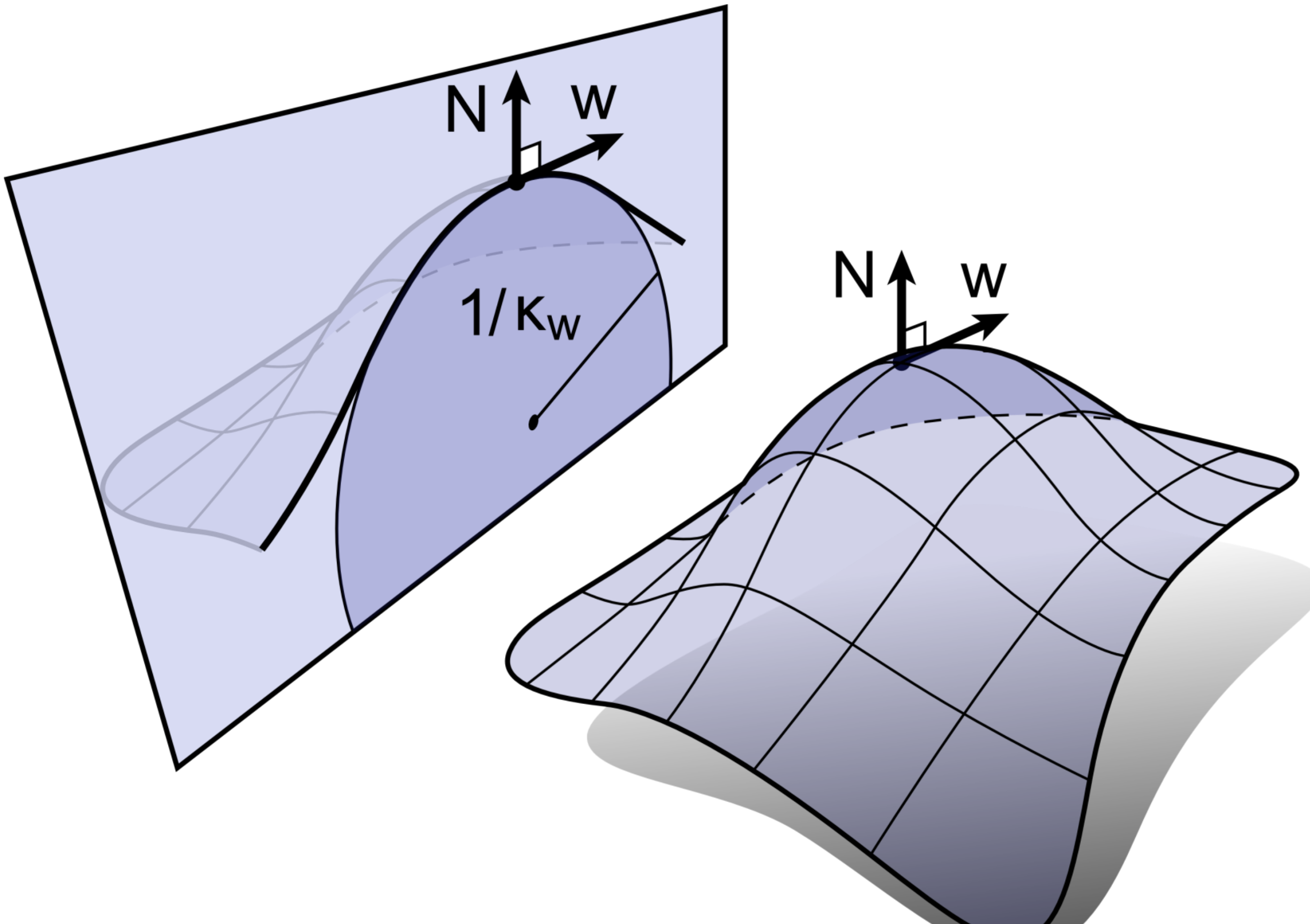
Idea #1: Normal Curvature

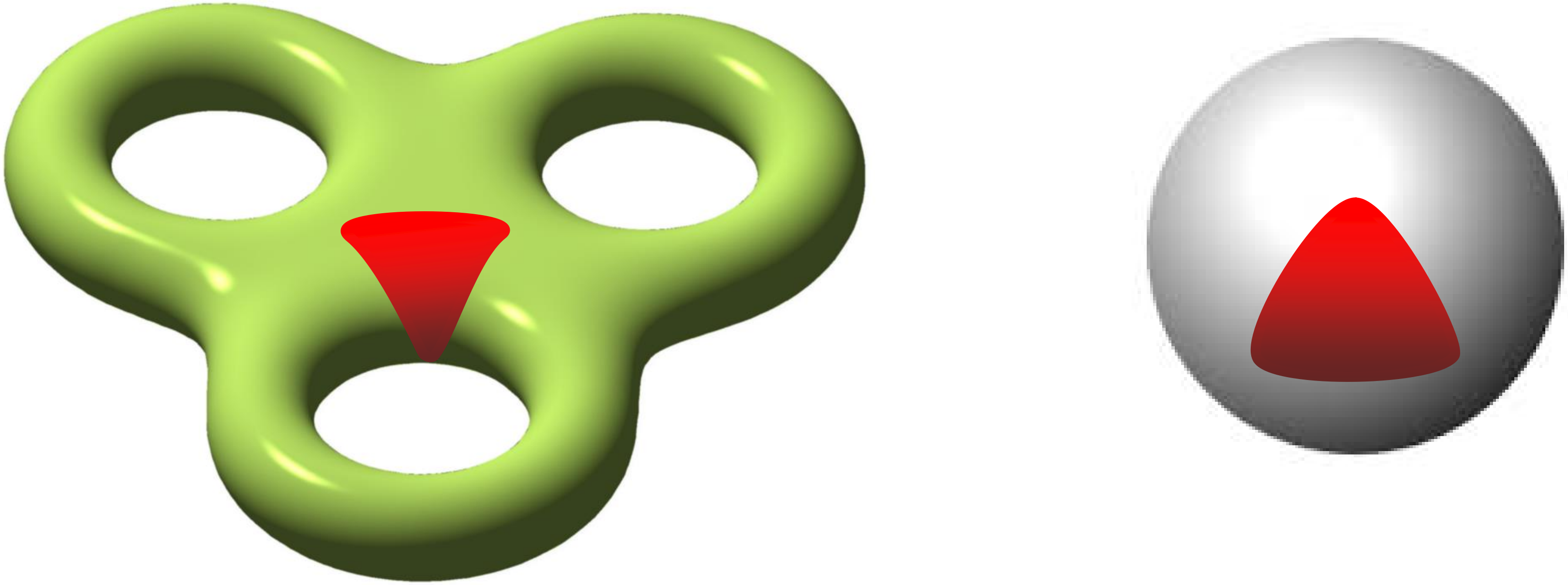


Mean Curvature H

Average normal curvature at point

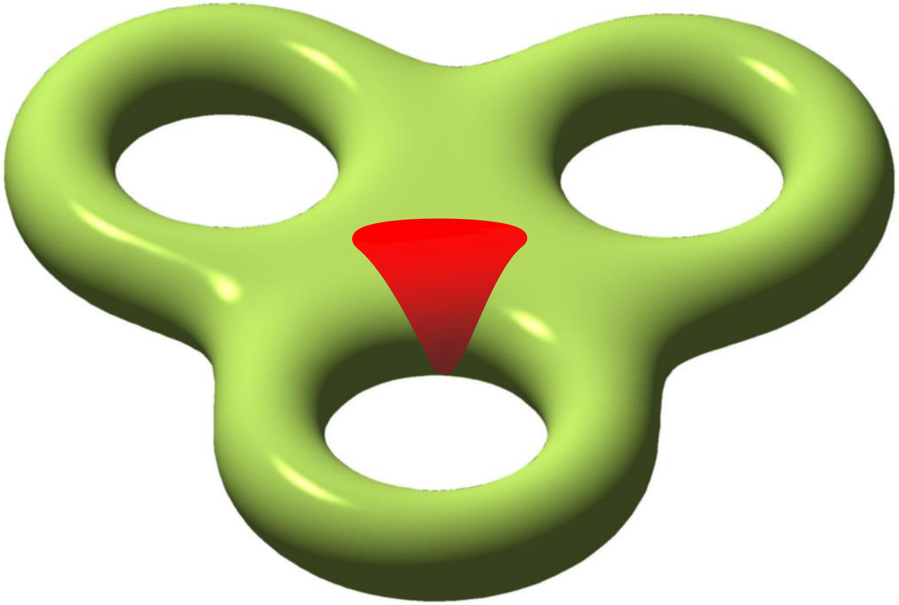


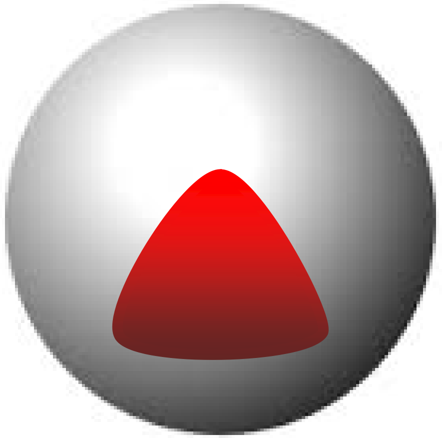
Idea #2: Look at Normals Again



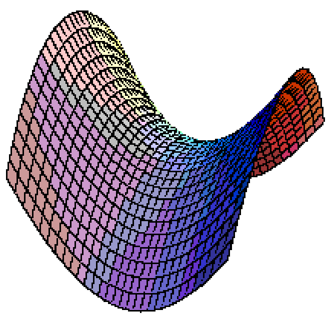
Idea #2: Look at Normals Again

Gaussian curvature $K = \frac{dA_N}{dA}$



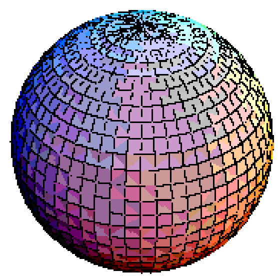


Mean and Gaussian Curvatue



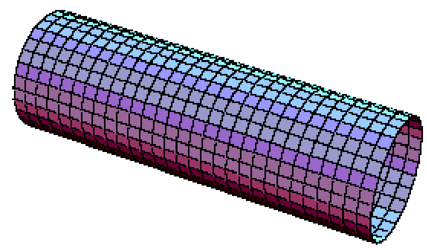
$K: -$

$H: 0$



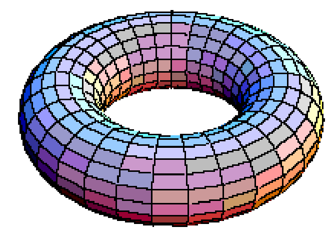
$K: +$

$H: +$



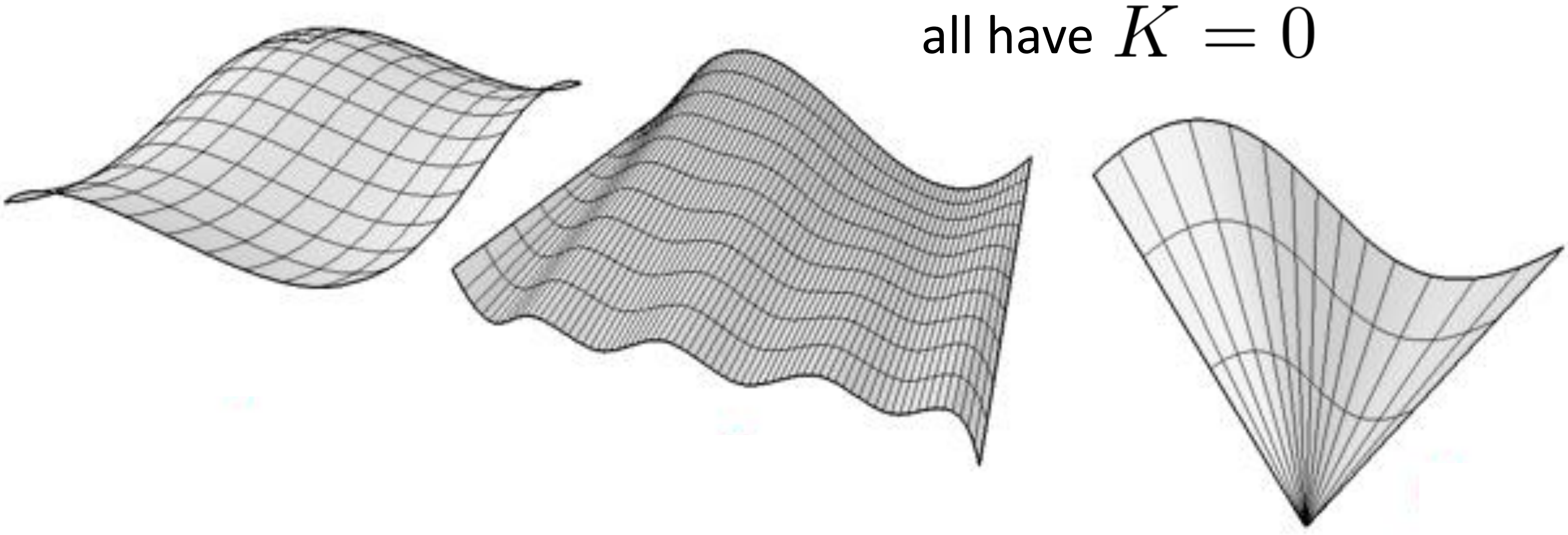
$K: 0$

$H: +$



Theorema Egregrium

Theorem (Gauss, deep): Gaussian curvature is an **isometry invariant**



Informativeness of Curvature

Theorem (easy): every curve can be reconstructed (up to rigid motions) from its curvature

Theorem (deep): every surface can be reconstructed (up to rigid motions) from its mean and Gaussian curvature

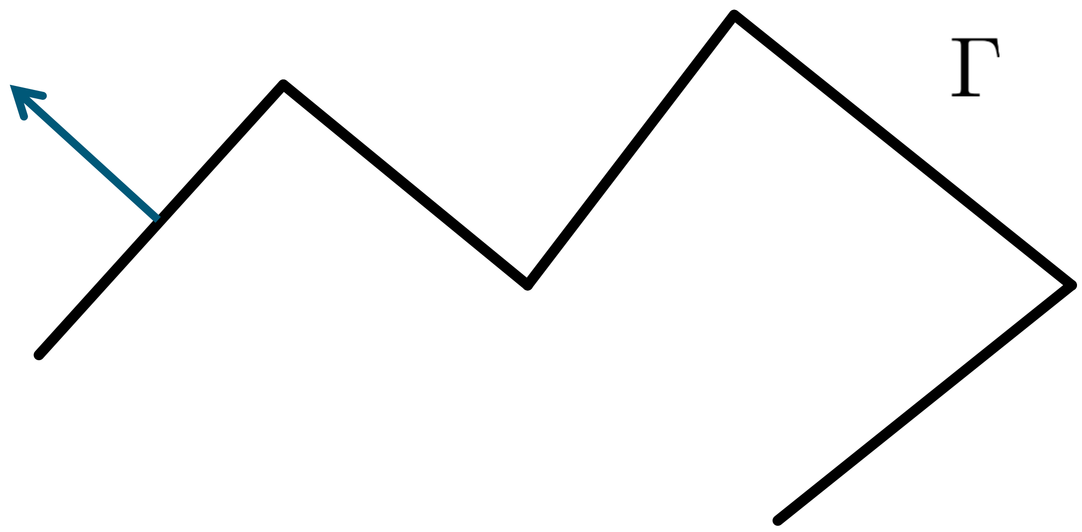
3D Analogues

Theorem [Gauss-Bonnet]: $\int K\,dA = 4\pi n$

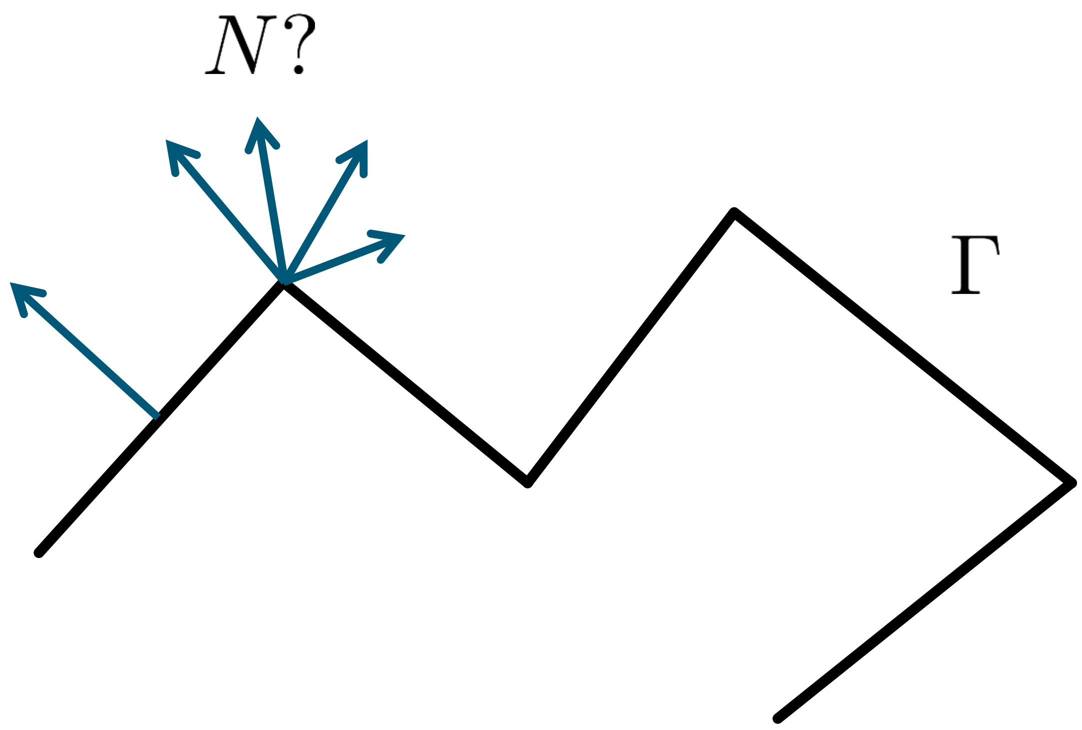
Theorem [Steiner]:

$$V_\epsilon = V + \epsilon A + \epsilon^2 \int H\,dA + \frac{1}{3}\epsilon^3 \int K\,dA$$

Discrete Curve

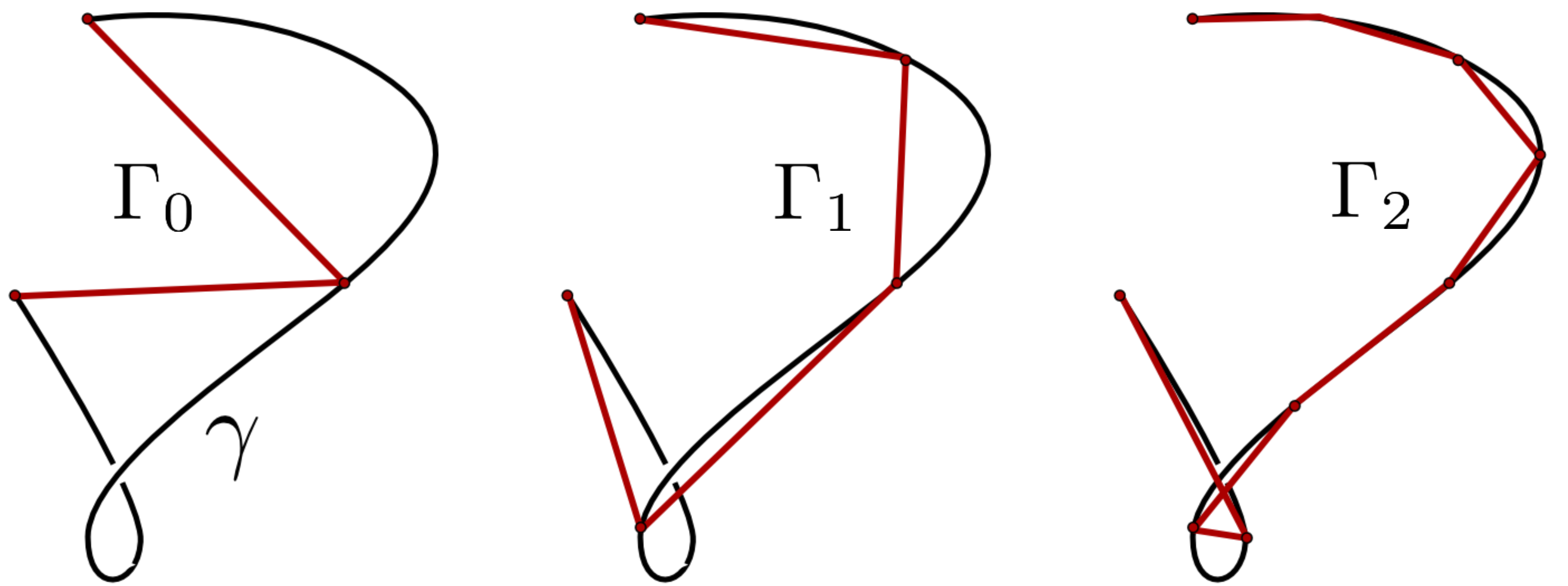


Discrete Curve



How do we Discretize Geometry?

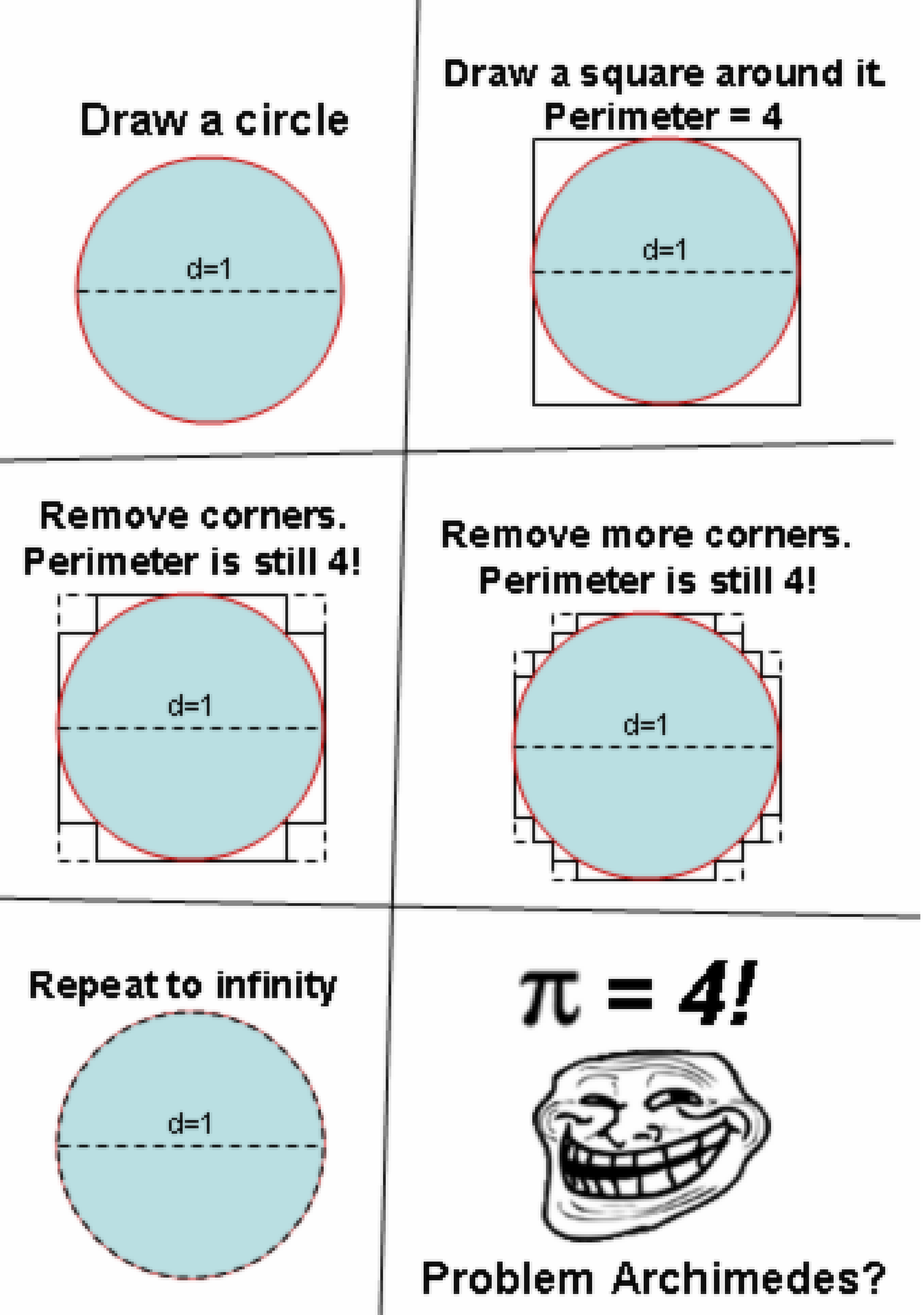
Option 1: Γ is not the "real curve." It approximates some smooth limit curve.



How do we Discretize Geometry?

Option 1: Γ is not the “real curve.” It approximates some smooth limit curve.

What is the refinement rule $\Gamma_i \to \Gamma_{i+1}$?



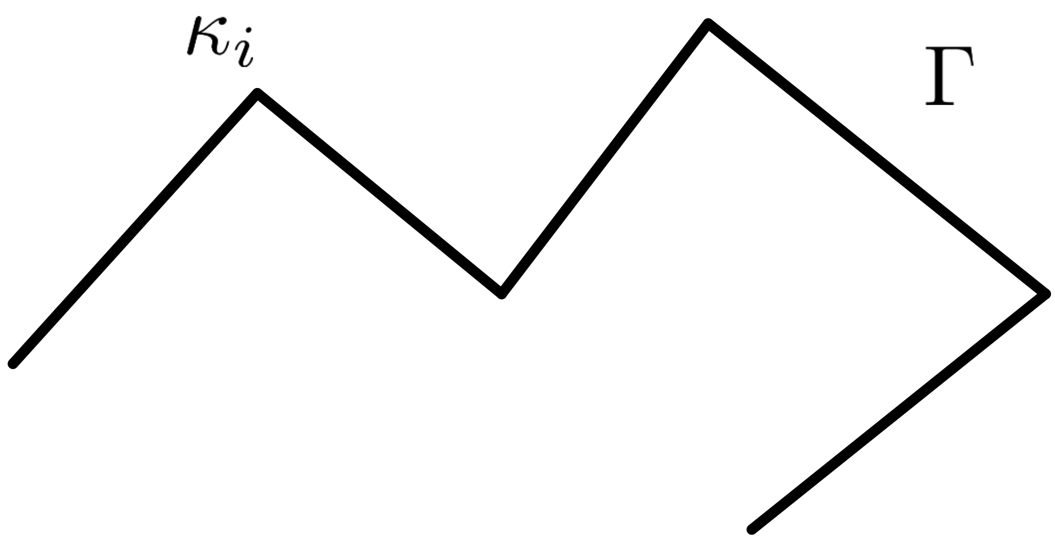
Internet "proof" that $\pi = 4$

How do we discretize geometry?

Option 2: Γ **is** the “real curve”! Construct geometry axiomatically

Get the right answer at every level of refinement

How do we discretize curvature?



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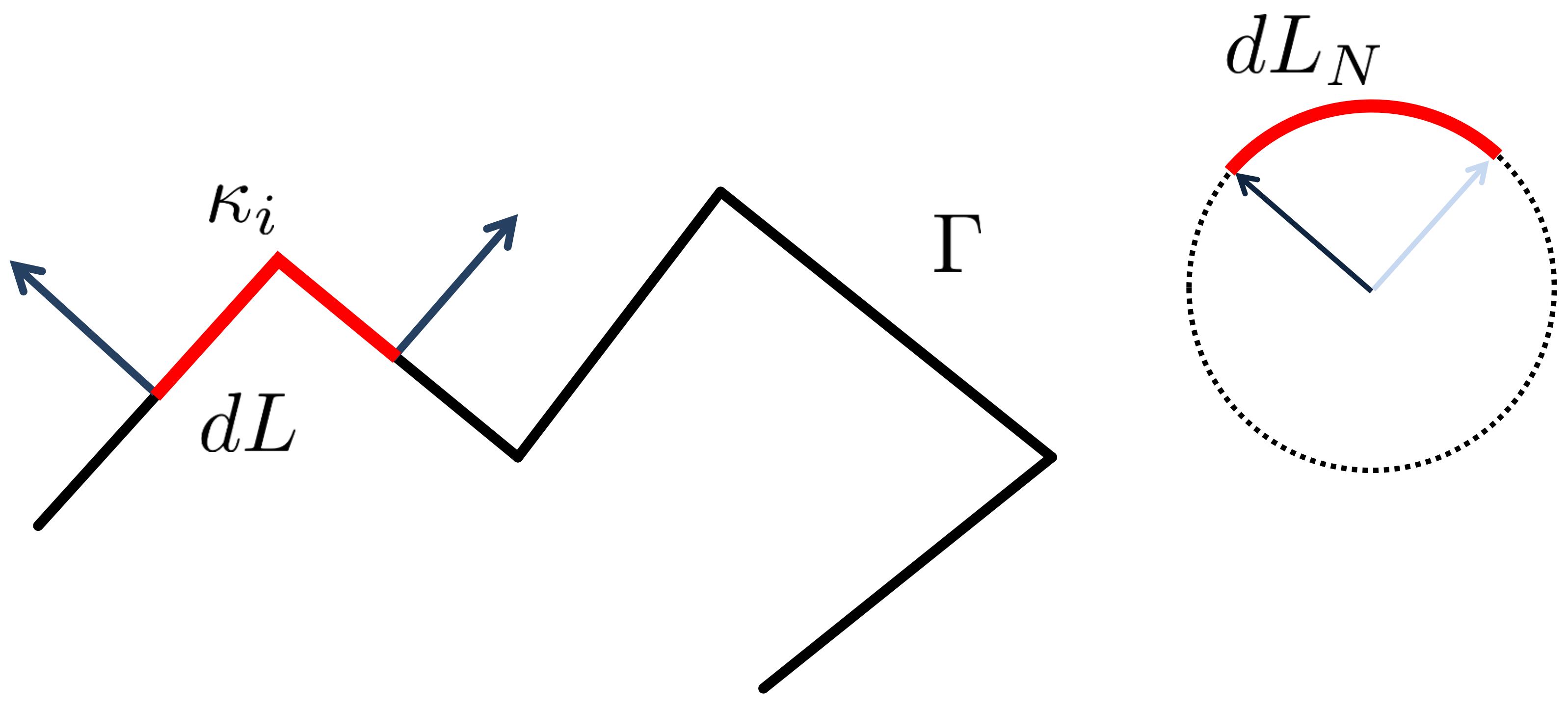
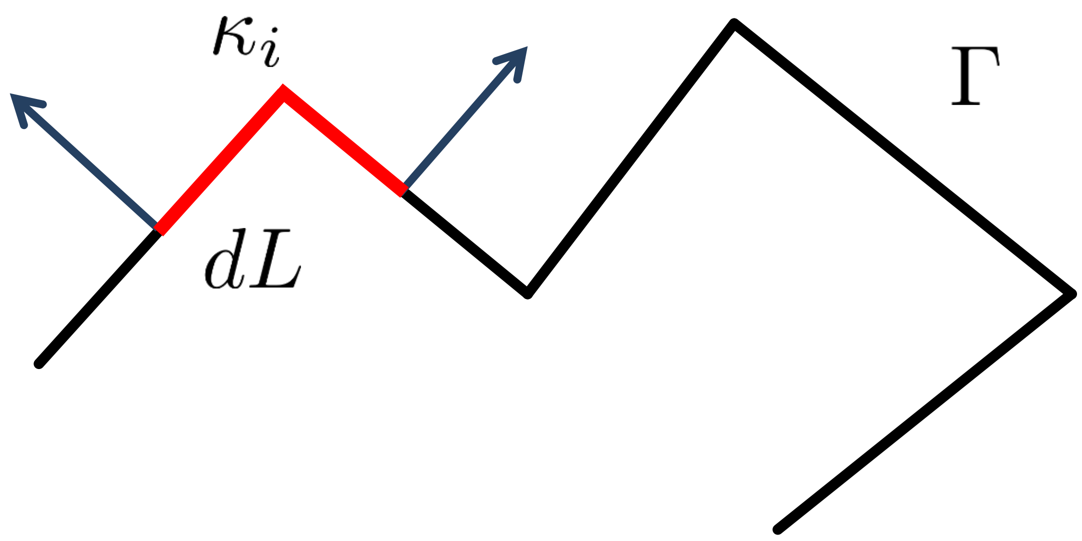
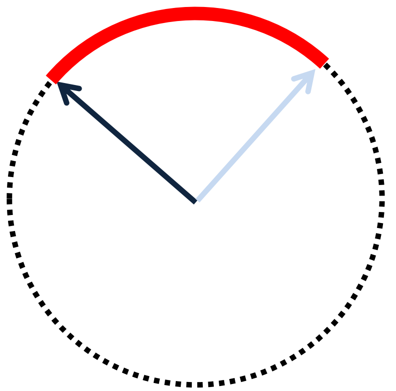


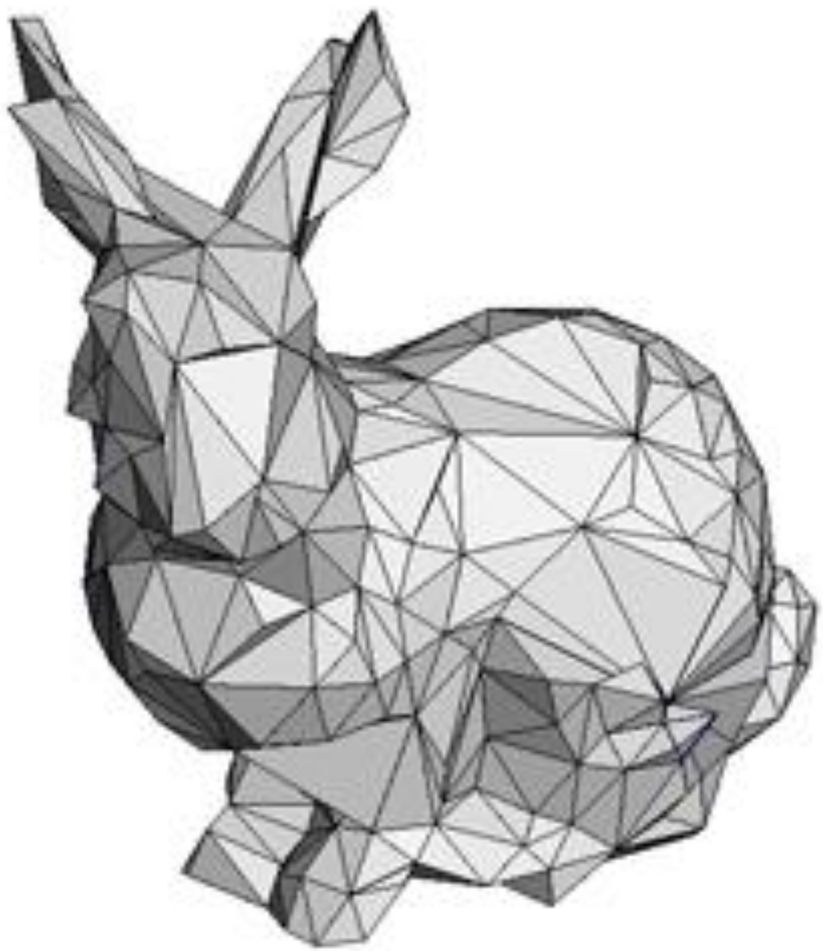
How do we discretize curvature?

$$\kappa_i = \frac{2\angle\left(p_{i+1} - p_i, p_i - p_{i-1}\right)}{\|p_{i+1} - p_i\| + \|p_i - p_{i-1}\|}$$

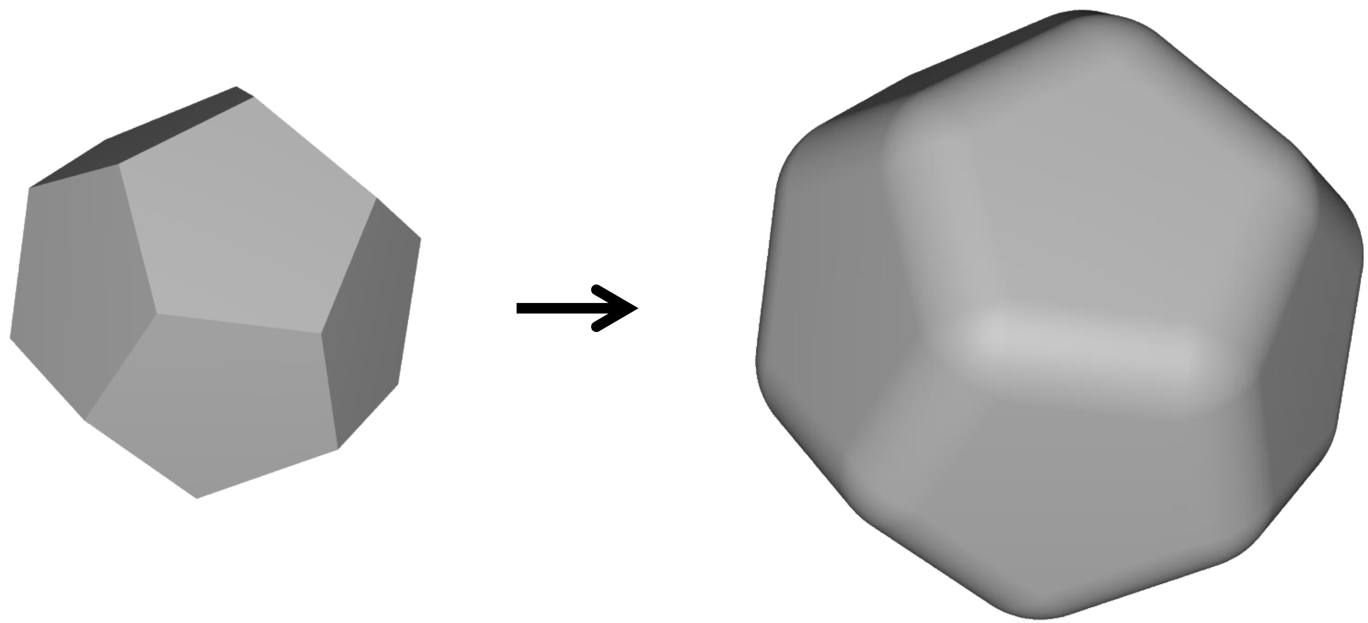




Discrete Surface



Discrete Inflation Theorem



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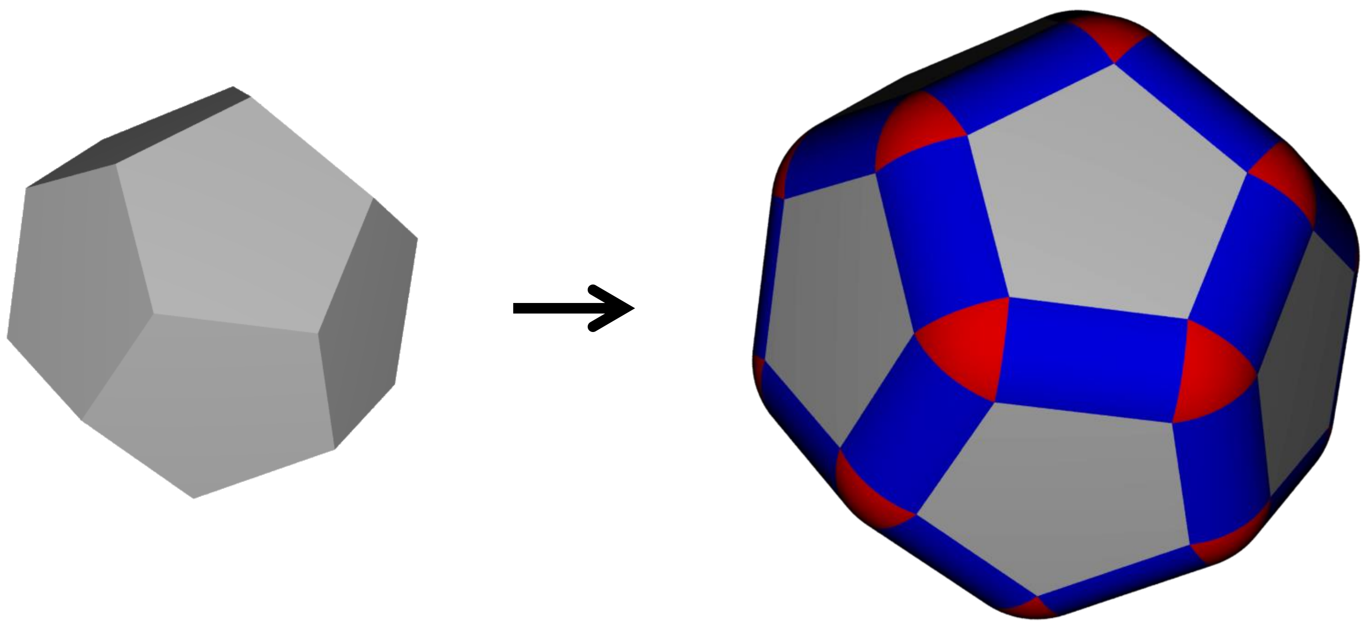
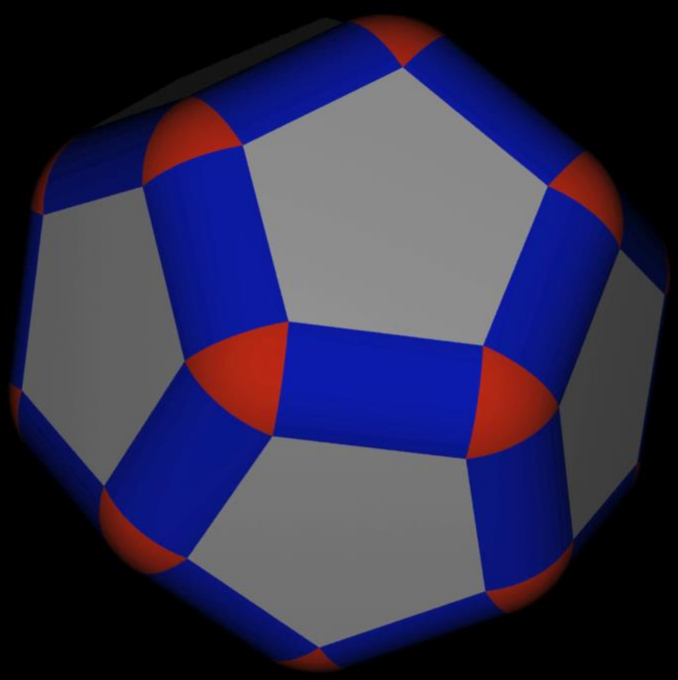


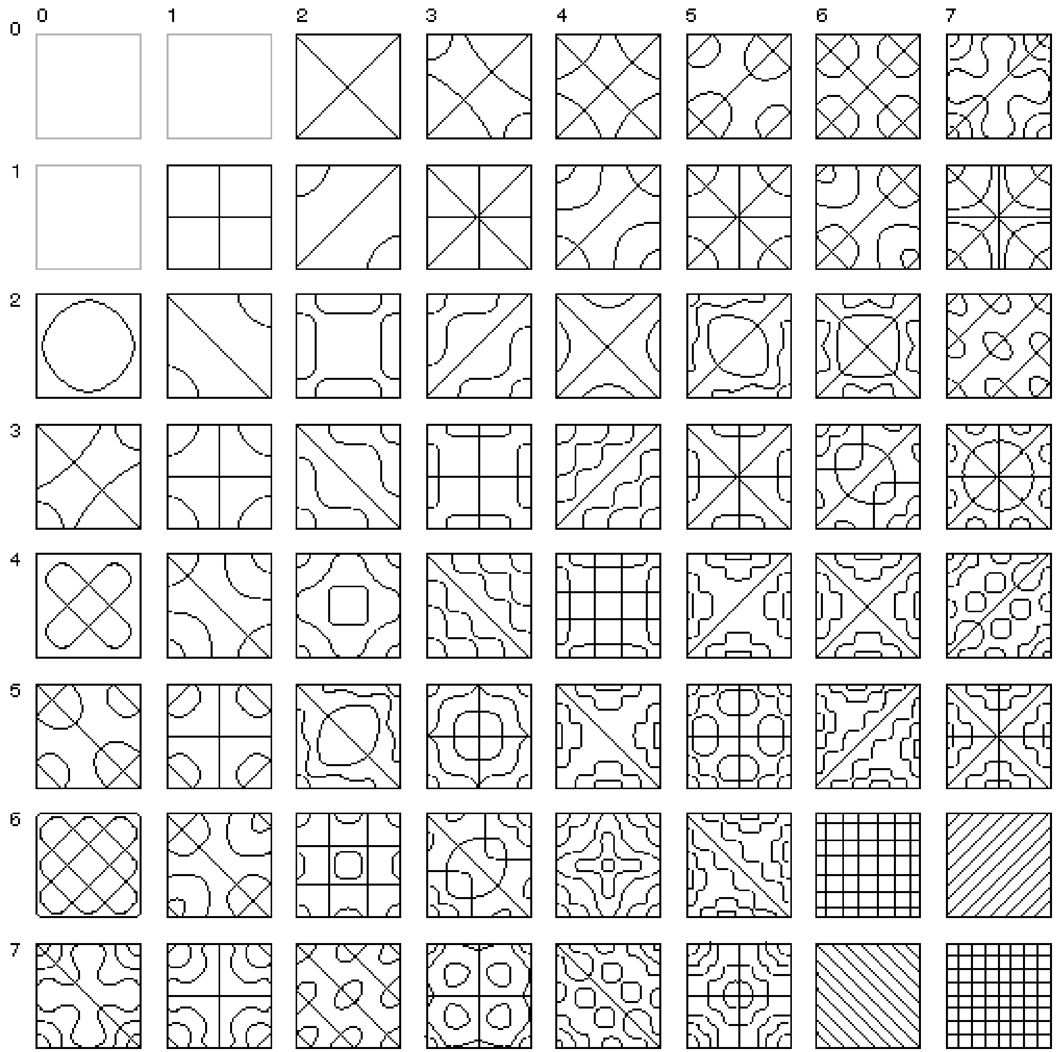
Discrete Gauss-Bonnet



Chladni Plates

Ernst Chladni

Isolines of Square Plate



Chladni Plates

Properties of plate energy:

- Stretching negligible
- Uniform, local & isotropic
- Zero for flat plate
- Same in both directions

Sophie Germain

Chladni Plates

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Sophie Germain

Low-order approximation: $E \propto \int H^2 \, dA$