Announcements

- Blackboard: view your grades and feedback on assignments.
- Typically can expect Pset grades by 1 week after deadline.
Today

- **Last time: Games with uncertainty**
  - Expectimax search
  - Mixed layer and multi-agent games
  - Defining utilities
  - Rational preferences
  - Human rationality, risk, and money

- **Today: Probability**
Recall: Rational Preferences

Preferences of a rational agent must obey constraints.

- The axioms of rationality:
  
  **Orderability**
  
  \[(A \succ B) \lor (B \succ A) \lor (A \sim B)\]
  
  **Transitivity**
  
  \[(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\]
  
  **Continuity**
  
  \[A \succ B \succ C \Rightarrow \exists p \ [p, A; 1 - p, C] \sim B\]
  
  **Substitutability**
  
  \[A \sim B \Rightarrow [p, A; 1 - p, C] \sim [p, B; 1 - p, C]\]
  
  **Monotonicity**
  
  \[A \succ B \Rightarrow (p \geq q \iff [p, A; 1 - p, B] \succeq [q, A; 1 - q, B])\]

- Theorem: Rational preferences imply behavior describable as maximization of expected utility.
Recall: MEU Principle

- **Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]**
  - Given any preferences satisfying these constraints, there exists a real-valued function $U$ such that:

  \[
  U(A) \geq U(B) \iff A \succeq B
  \]

  \[
  U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)
  \]

  - i.e., values assigned by $U$ preserve preferences of both prizes and lotteries!

- **Maximum expected utility (MEU) principle:**
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
    - E.g., a lookup table for perfect tic-tactoe, reflex vacuum cleaner
Recall: Money

- Money does not behave as a utility function, but we can talk about the utility of having money (or being in debt)

- Given a lottery \( L = [p, X; (1-p), Y] \)
  - The expected monetary value \( EMV(L) \) is \( pX + (1-p)Y \)
  - \( U(L) = pU(X) + (1-p)U(Y) \)
  - Typically, \( U(L) < U(EMV(L)) \): why?

- In this sense, people are risk-averse
- When deep in debt, we are risk-prone
Example: Insurance

- Consider the lottery $[0.5,1000; 0.5,0]$
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There’s an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!
Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8,$4k; 0.2,$0]
  - B: [1.0,$3k; 0.0,$0]
  - C: [0.2,$4k; 0.8,$0]
  - D: [0.25,$3k; 0.75,$0]

- Most people prefer B > A, C > D

- But if $U(0) = 0$, then
  - B > A $\Rightarrow$ $U(3k) > 0.8 U(4k)$
  - C > D $\Rightarrow$ 0.8 $U(4k) > U(3k)$
Today

- Last time: Games with uncertainty
  - Expectimax search
  - Mixed layer and multi-agent games
  - Defining utilities
  - Rational preferences
  - Human rationality, risk, and money

- Today: Probability
Need for probability

- Search and planning
- Probabilistic reasoning (Part II of course)
  - Diagnosis
  - Speech recognition
  - Tracking objects
  - Robot mapping
  - Genetics
  - Error correcting codes
  - …lots more!
- Machine learning (Part III of course)
Topics

- **Probability**
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes’ Rule
  - Inference
  - Independence

- You’ll need all this stuff A LOT in subsequent weeks, so make sure you go over it now!
Inference in Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green
- Sensors are noisy, but we know $P(\text{Color} \mid \text{Distance})$

<table>
<thead>
<tr>
<th>$P(\text{red} \mid 3)$</th>
<th>$P(\text{orange} \mid 3)$</th>
<th>$P(\text{yellow} \mid 3)$</th>
<th>$P(\text{green} \mid 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.15</td>
<td>0.5</td>
<td>0.3</td>
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</tbody>
</table>
Inference in Ghostbusters

<table>
<thead>
<tr>
<th>0.11</th>
<th>0.11</th>
<th>0.11</th>
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</thead>
<tbody>
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<td>0.11</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>0.17</th>
<th>0.10</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>0.17</td>
<td>0.10</td>
</tr>
<tr>
<td>&lt;0.01</td>
<td>0.09</td>
<td>0.17</td>
</tr>
</tbody>
</table>
Uncertainty

- **General situation:**
  - **Observed variables (evidence):** Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables:** Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model:** Agent knows something about how the known variables relate to the unknown variables

- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge
Random Variables

- A **random variable** is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - D = How long will UT delay for winter weather?
  - L = Where is the ghost?

- We denote random variables with capital letters

- Random variables have **domains**
  - R in {true, false} (sometimes write as {+r, ¬r})
  - D in [0, 8)
  - L in possible locations, maybe {(0,0), (0,1), …}
Probability Distributions

- Unobserved random variables have distributions

\[ P(T) \]

<table>
<thead>
<tr>
<th>T</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>warm</td>
<td>0.5</td>
</tr>
<tr>
<td>cold</td>
<td>0.5</td>
</tr>
</tbody>
</table>

\[ P(W) \]

<table>
<thead>
<tr>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>sun</td>
<td>0.6</td>
</tr>
<tr>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>fog</td>
<td>0.3</td>
</tr>
<tr>
<td>meteor</td>
<td>0.0</td>
</tr>
</tbody>
</table>

- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number

\[ P(W = rain) = 0.1 \quad P(rain) = 0.1 \]

- Must have: \( \forall x \; P(x) \geq 0 \) \quad \( \sum_x P(x) = 1 \)
Joint Distributions

- A *joint distribution* over a set of random variables: \( X_1, X_2, \ldots X_n \)
  specifies a real number for each assignment (or *outcome*):

\[
P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) \quad P(x_1, x_2, \ldots x_n)
\]

- Size of distribution if \( n \) variables with domain sizes \( d \)?

- Must obey:
  \[
P(x_1, x_2, \ldots x_n) \geq 0
  \]
  \[
  \sum_{(x_1,x_2,\ldots x_n)} P(x_1, x_2, \ldots x_n) = 1
  \]

- For all but the smallest distributions, impractical to write out

<p>| | | |</p>
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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>W</td>
<td>P</td>
</tr>
<tr>
<td>hot</td>
<td>sun</td>
<td>0.4</td>
</tr>
<tr>
<td>hot</td>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>cold</td>
<td>sun</td>
<td>0.2</td>
</tr>
<tr>
<td>cold</td>
<td>rain</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables

- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
    - *Normalized:* sum to 1.0
    - Ideally: only certain variables directly interact

<table>
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<td>hot</td>
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<td>0.4</td>
</tr>
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<td>rain</td>
<td>0.1</td>
</tr>
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<td>cold</td>
<td>sun</td>
<td>0.2</td>
</tr>
<tr>
<td>cold</td>
<td>rain</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Events

- An event is a set $E$ of outcomes

\[ P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n) \]

- From a joint distribution, we can calculate the probability of any event
  - Probability that it’s hot AND sunny?
  - Probability that it’s hot?
  - Probability that it’s hot OR sunny?

- Typically, the events we care about are partial assignments, like $P(T=\text{hot})$
Quiz

1. $P(+x, +y)$?

2. $P(+x)$?

3. $P(-y \text{ OR } +x)$ ?

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>+x</td>
<td>+y</td>
<td>0.2</td>
</tr>
<tr>
<td>+x</td>
<td>-y</td>
<td>0.3</td>
</tr>
<tr>
<td>-x</td>
<td>+y</td>
<td>0.4</td>
</tr>
<tr>
<td>-x</td>
<td>-y</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables.
- Marginalization (summing out): Combine collapsed rows by adding.

\[
P(T, W)
\]

<table>
<thead>
<tr>
<th>T</th>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>hot</td>
<td>sun</td>
<td>0.4</td>
</tr>
<tr>
<td>hot</td>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>cold</td>
<td>sun</td>
<td>0.2</td>
</tr>
<tr>
<td>cold</td>
<td>rain</td>
<td>0.3</td>
</tr>
</tbody>
</table>

\[
P(T) = \sum_s P(t, s)
\]

<table>
<thead>
<tr>
<th>T</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>hot</td>
<td>0.5</td>
</tr>
<tr>
<td>cold</td>
<td>0.5</td>
</tr>
</tbody>
</table>

\[
P(W)
\]

<table>
<thead>
<tr>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>sun</td>
<td>0.6</td>
</tr>
<tr>
<td>rain</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\[
P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)
\]
Quiz: marginal distributions

\[ P(X, Y) \]

\[
\begin{array}{ccc}
\text{X} & \text{Y} & \text{P} \\
+\text{x} & +\text{y} & 0.2 \\
+\text{x} & -\text{y} & 0.3 \\
-\text{x} & +\text{y} & 0.4 \\
-\text{x} & -\text{y} & 0.1 \\
\end{array}
\]

\[
P(x) = \sum_{y} P(x, y)
\]

\[
P(y) = \sum_{x} P(x, y)
\]
Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the definition of a conditional probability

\[ P(a|b) = \frac{P(a, b)}{P(b)} \]

<table>
<thead>
<tr>
<th>T</th>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>hot</td>
<td>sun</td>
<td>0.4</td>
</tr>
<tr>
<td>hot</td>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>cold</td>
<td>sun</td>
<td>0.2</td>
</tr>
<tr>
<td>cold</td>
<td>rain</td>
<td>0.3</td>
</tr>
</tbody>
</table>

\[ P(W = r|T = c) = ???? \]
Quiz: conditional probabilities

\[
P(X, Y)
\]

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>+x</td>
<td>+y</td>
<td>0.2</td>
</tr>
<tr>
<td>+x</td>
<td>-y</td>
<td>0.3</td>
</tr>
<tr>
<td>-x</td>
<td>+y</td>
<td>0.4</td>
</tr>
<tr>
<td>-x</td>
<td>-y</td>
<td>0.1</td>
</tr>
</tbody>
</table>

- \( P(+x \mid +y) \) ?
- \( P(-x \mid +y) \) ?
- \( P(-y \mid +x) \) ?
Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others.

| $P(W|T = \text{hot})$ | $P(W|T = \text{cold})$ |
|-----------------------|-----------------------|
| **W** | **P** | **W** | **P** |
| sun     | 0.8    | sun     | 0.4    |
| rain    | 0.2    | rain    | 0.6    |

<table>
<thead>
<tr>
<th>$P(T, W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
</tr>
<tr>
<td>hot</td>
</tr>
<tr>
<td>hot</td>
</tr>
<tr>
<td>cold</td>
</tr>
<tr>
<td>cold</td>
</tr>
</tbody>
</table>
Computing conditional probabilities

\[ P(W = s | T = c) = \frac{P(W = s, T = c)}{P(T = c)} \]
\[ = \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \]
\[ = \frac{0.2}{0.2 + 0.3} = 0.4 \]

\[ P(W = r | T = c) = \frac{P(W = r, T = c)}{P(T = c)} \]
\[ = \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \]
\[ = \frac{0.3}{0.2 + 0.3} = 0.6 \]
Normalization Trick

- A trick to get a whole conditional distribution at once:
  1. Select the joint probabilities matching the evidence
  2. Normalize the selection (make it sum to one)

\[
P(T, W)
\]

<table>
<thead>
<tr>
<th>T</th>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>hot</td>
<td>sun</td>
<td>0.4</td>
</tr>
<tr>
<td>hot</td>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>cold</td>
<td>sun</td>
<td>0.2</td>
</tr>
<tr>
<td>cold</td>
<td>rain</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Why does this work? Sum of selection is \( P(\text{evidence}) \)! (\( P(c) \) here)

\[
P(x_1 | x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}
\]
Quiz: normalization trick

\[ P(X | Y=-y) ? \]

<table>
<thead>
<tr>
<th>( P(X, Y) )</th>
<th>( X )</th>
<th>( Y )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+x</td>
<td>+y</td>
<td>0.2</td>
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<tr>
<td></td>
<td>+x</td>
<td>-y</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>-x</td>
<td>+y</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>-x</td>
<td>-y</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**Select** the joint probabilities matching the evidence

**Normalize** the selection (make it sum to one)
Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - \( P(\text{on time} \mid \text{no reported accidents}) = 0.90 \)
  - These represent the agent’s beliefs given the evidence

- Probabilities change with new evidence:
  - \( P(\text{on time} \mid \text{no accidents, 5 a.m.}) = 0.95 \)
  - \( P(\text{on time} \mid \text{no accidents, 5 a.m., raining}) = 0.80 \)
  - Observing new evidence causes beliefs to be updated
Inference by Enumeration

- $P(\text{sun})$?

- $P(\text{sun} | \text{winter})$?

- $P(\text{sun} | \text{winter}, \text{hot})$?

<table>
<thead>
<tr>
<th>S</th>
<th>T</th>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>summer</td>
<td>hot</td>
<td>sun</td>
<td>0.30</td>
</tr>
<tr>
<td>summer</td>
<td>hot</td>
<td>rain</td>
<td>0.05</td>
</tr>
<tr>
<td>summer</td>
<td>cold</td>
<td>sun</td>
<td>0.10</td>
</tr>
<tr>
<td>summer</td>
<td>cold</td>
<td>rain</td>
<td>0.05</td>
</tr>
<tr>
<td>winter</td>
<td>hot</td>
<td>sun</td>
<td>0.10</td>
</tr>
<tr>
<td>winter</td>
<td>hot</td>
<td>rain</td>
<td>0.05</td>
</tr>
<tr>
<td>winter</td>
<td>cold</td>
<td>sun</td>
<td>0.15</td>
</tr>
<tr>
<td>winter</td>
<td>cold</td>
<td>rain</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Inference by Enumeration

- **General case:**
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: $Q$
  - Hidden variables: $H_1 \ldots H_r$

  $\begin{cases}
  X_1, X_2, \ldots X_n \\
  \text{All variables}
  \end{cases}$

- **We want:** $P(Q|e_1 \ldots e_k)$

  1. Select the entries consistent with the evidence
  2. Sum out H to get joint of Query and evidence:

  $$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} P(Q, h_1 \ldots h_r, e_1 \ldots e_k)$$

  $X_1, X_2, \ldots X_n$

  3. Normalize

  $$Z = \sum_q P(Q, e_1, \ldots, e_k)$$

  $$P(Q|e_1, \ldots, e_k) = \frac{1}{Z} \sum_q P(Q, e_1, \ldots, e_k)$$

* Works fine with multiple query variables, too
The Product Rule

- Sometimes have conditional distributions but want the joint

\[ P(x|y) = \frac{P(x, y)}{P(y)} \quad \text{↔} \quad P(x, y) = P(x|y)P(y) \]

- Example:

<table>
<thead>
<tr>
<th>P(W)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>P</td>
<td></td>
</tr>
<tr>
<td>sun</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>rain</td>
<td>0.2</td>
<td></td>
</tr>
</tbody>
</table>

| P(D|W) |   |   |   |
|-------|---|---|---|
| D     | W | P |
| wet   | sun | 0.1 |
| dry   | sun | 0.9 |
| wet   | rain| 0.7 |
| dry   | rain| 0.3 |

<table>
<thead>
<tr>
<th>P(D, W)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>W</td>
<td>P</td>
<td></td>
</tr>
<tr>
<td>wet</td>
<td>sun</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dry</td>
<td>sun</td>
<td></td>
<td></td>
</tr>
<tr>
<td>wet</td>
<td>rain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dry</td>
<td>rain</td>
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</table>
The Chain Rule

More generally, can always write any joint distribution as an incremental product of conditional distributions

\[ P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \]

\[ P(x_1, x_2, \ldots x_n) = \prod_{i} P(x_i|x_1 \ldots x_{i-1}) \]

Why is this always true?
Bayes’ Rule

- Two ways to factor a joint distribution over two variables:

\[ P(x, y) = P(x|y)P(y) = P(y|x)P(x) \]

- Dividing, we get:

\[ P(x|y) = \frac{P(y|x)}{P(y)}P(x) \]

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we’ll see later

- In the running for most important AI equation!
Inference with Bayes’ Rule

- Example: Diagnostic probability from causal probability:

\[ P(\text{Cause}|\text{Effect}) = \frac{P(\text{Effect}|\text{Cause})P(\text{Cause})}{P(\text{Effect})} \]

- Example:
  - m is meningitis, s is stiff neck
  - \( P(s|m) = 0.8 \)
  - \( P(m) = 0.0001 \)
  - \( P(s) = 0.1 \)

\[ P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008 \]

- Note: posterior probability of meningitis still very small
- Note: you should still get stiff necks checked out! Why?
Example: learning skin colors

- We can represent a class-conditional density using a histogram (a “non-parametric” distribution)
Example: learning skin colors

- We can represent a class-conditional density using a histogram (a “non-parametric” distribution)

Now we get a new image, and want to label each pixel as skin or non-skin.

What’s the probability we care about to do skin detection?
Example: learning skin colors

\[ P(skin \mid x) = \frac{P(x \mid skin)P(skin)}{P(x)} \]

\[ P(skin \mid x) \propto P(x \mid skin)P(skin) \]

Where might the prior come from?
Example: learning skin colors

Now for every pixel in a new image, we can estimate probability that it is generated by skin.

Classify pixels based on these probabilities

- if \( p(\text{skin} | \mathbf{x}) > \theta \), classify as skin
- if \( p(\text{skin} | \mathbf{x}) < \theta \), classify as not skin

Brighter pixels \( \rightarrow \) higher probability of being skin
Quiz: Bayes’ Rule

- What is $P(W \mid \text{dry})$?
Let’s say we have two distributions:

- Prior distribution over ghost location: \( P(G) \)
  - Let’s say this is uniform
- Sensor reading model: \( P(R \mid G) \)
  - Given: we know what our sensors do
  - \( R = \) reading color measured at \((1,1)\)
  - E.g. \( P(R = \text{yellow} \mid G=(1,1)) = 0.1 \)

We can calculate the posterior distribution \( P(G\mid r) \) over ghost locations given a reading using Bayes’ rule:

\[
P(g\mid r) \propto P(r\mid g)P(g)
\]
Summary

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes’ Rule
  - Inference

- Next time:
  - Independence
  - Bayesian Networks