CS354 Computer Graphics
Vector and Affine Math

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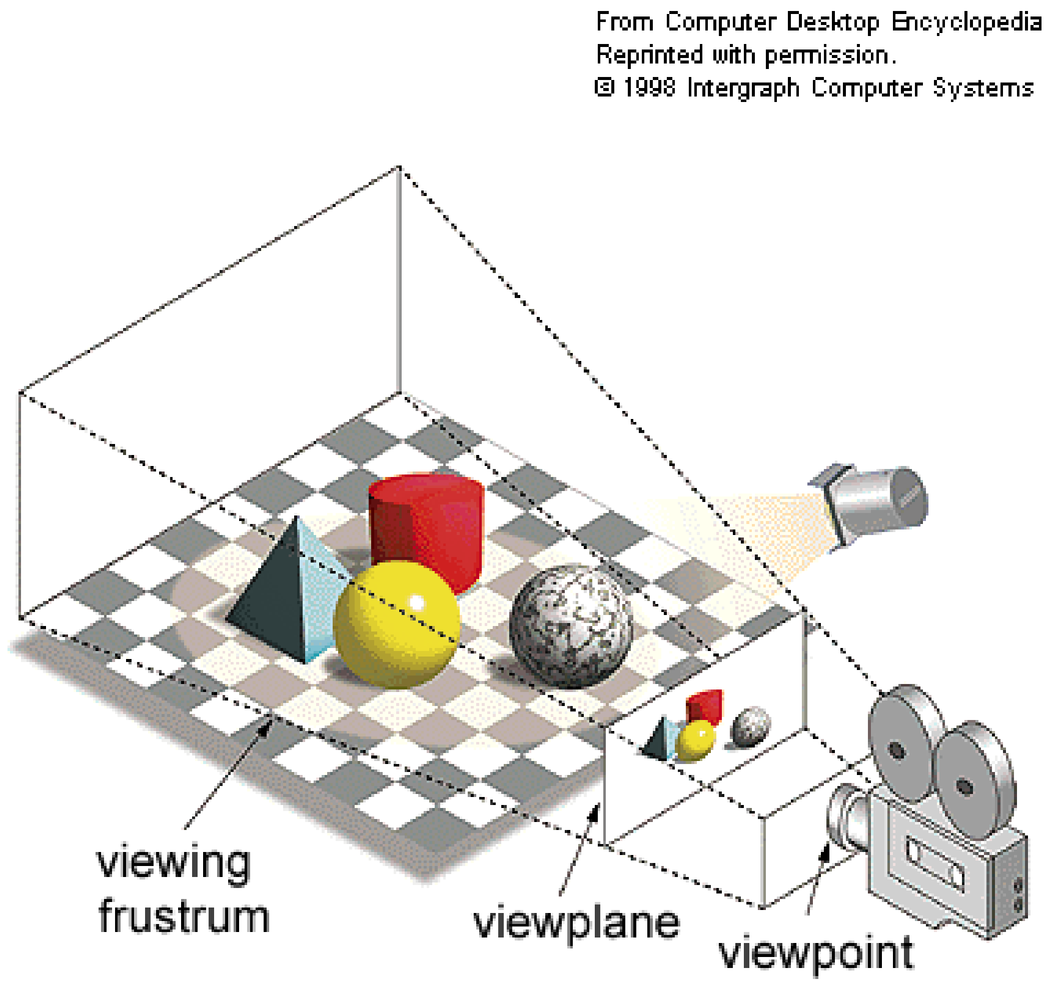
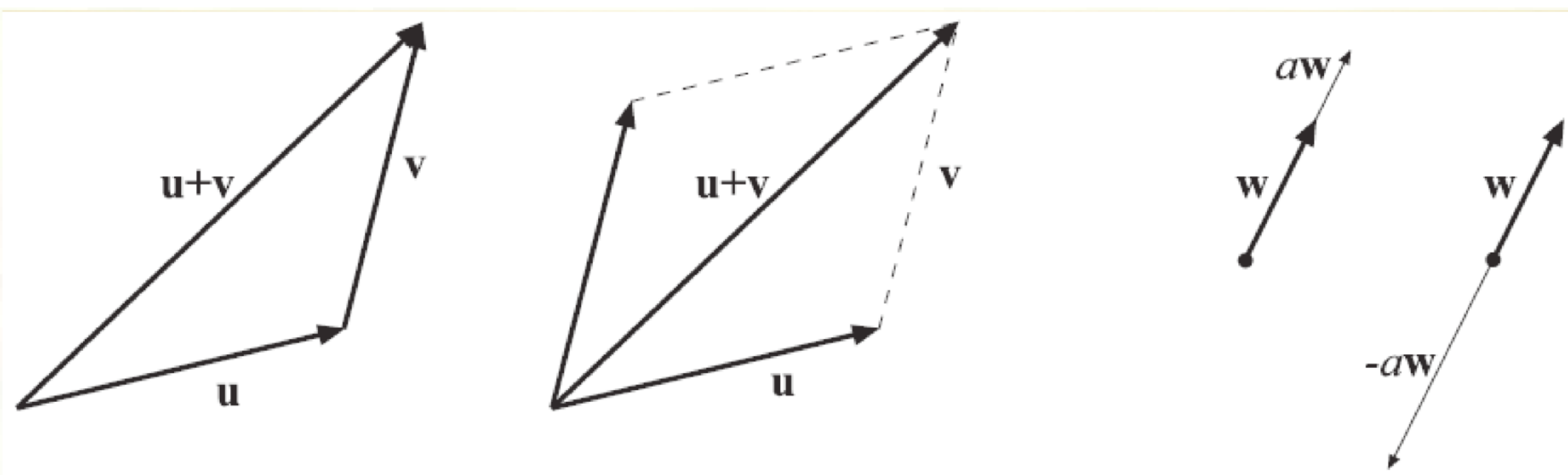
Graphics Pipeline



Vectors

- A vector is a direction and a magnitude
- Does NOT include a point of reference
- Usually thought of as an arrow in space
- Vectors can be added together and multiplied by scalars
- Zero vector has no length or direction

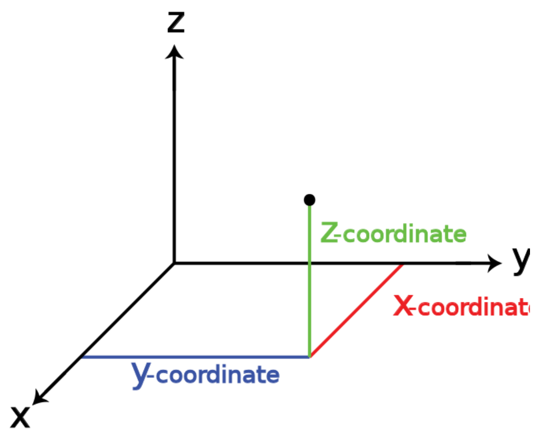


Vector Spaces

- Set of vectors
- Closed under the following operations
  - Vector addition
  - Scalar multiplication
  - Linear combinations
- Scalars come from some field F
  - e.g. real or complex numbers
- Linear independence
- Basis
- Dimension

Coordinate Representation

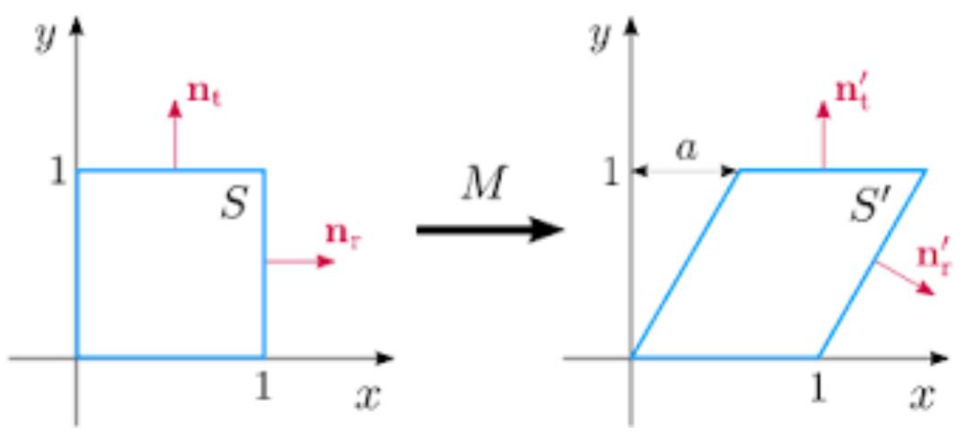
- Pick a basis, order the vectors in it, then all vectors in the space can be represented as sequences of coordinates, i.e. coefficients of the basis vectors, in order
- The most widely used represention is Cartersian 3-space
- There are row and column vectors, and we usually use column vectors



Linear Transformations

- Given vector spaces V and W
- A function f: V -> W is a linear map or linear transformation if

$$f(a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) = a_1 f(\mathbf{v}_1) + \ldots + a_m f(\mathbf{v}_m)$$



Transformation Representation

- Under the choices of basis, we can represent a 2-D transformation M by a matrix

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- It gives the relation of the coordiantes:

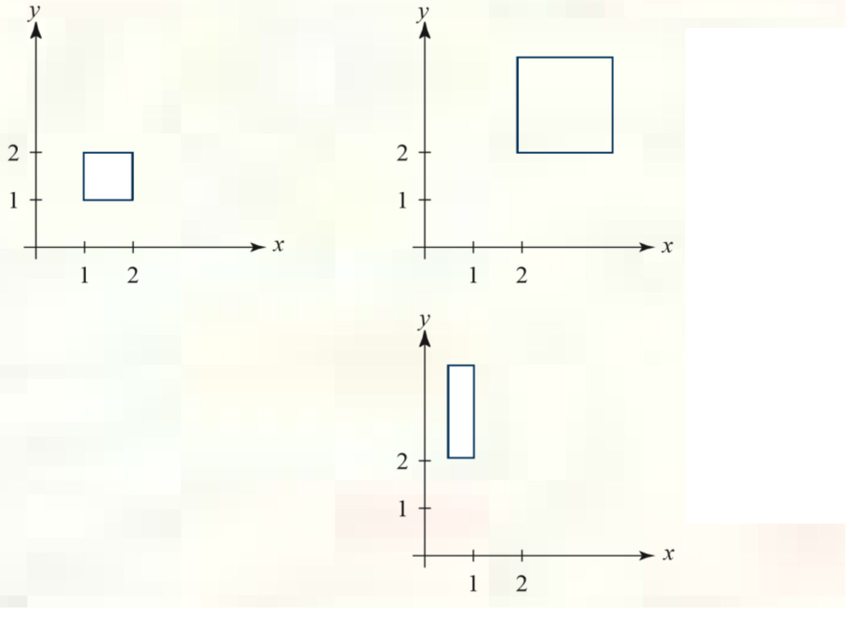
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Identity

- Suppose we choose a=d=1, b=c=0:
- Gives the identity matrix: [1 0;0 1]
- Doesn’t change anything

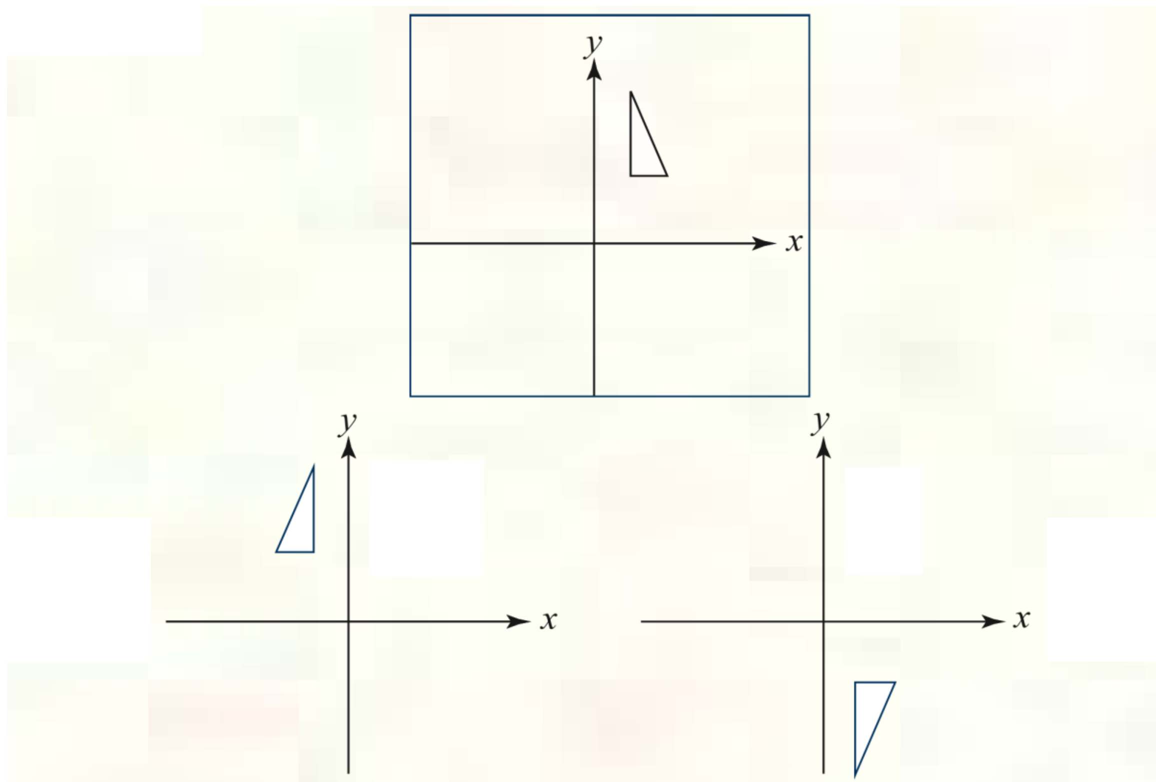
Scaling

- Suppose b=c=0, but let a and d take on any positive value
  - Gives a scaling matrix: [a 0; 0 d]



Reflection

- Suppose b=c=0, but let either a or d go negative
- Examples:



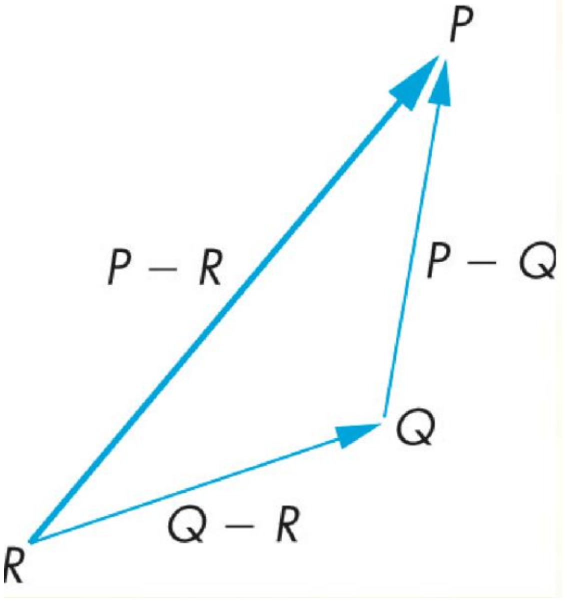
Limitations of the 2 x 2 matrix

- A 2 x 2 linear transformation matrix allows
  - Scaling
  - Rotation
  - Reflection
  - Shearing

- Q: What important operation does that leave out?

Points

- A point is a location in space
- Cannot be added or multiplied together
- Subtract two points to get the vector between them
- Points are not vectors



Affine transformations

- In order to incorporate the idea that both the basis and the origin can change, we augment the linear space u, w with an origin t
- Note that while u and w are basis vectors, the origin t is a point
- We call u, w, and t (basis and origin) a frame for an affine space
- Then, we can represent a change of frame as

$$\mathbf{p}' = x \cdot \mathbf{u} + y \cdot \mathbf{w} + \mathbf{t}$$

- This change of frame is also known as an affine transformation

Basic Vector Arithmetic

$$\mathbf{u} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} r + x \\ s + y \\ t + z \end{bmatrix} \quad a\mathbf{v} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$$

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2} \quad norm(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Parametric line segment

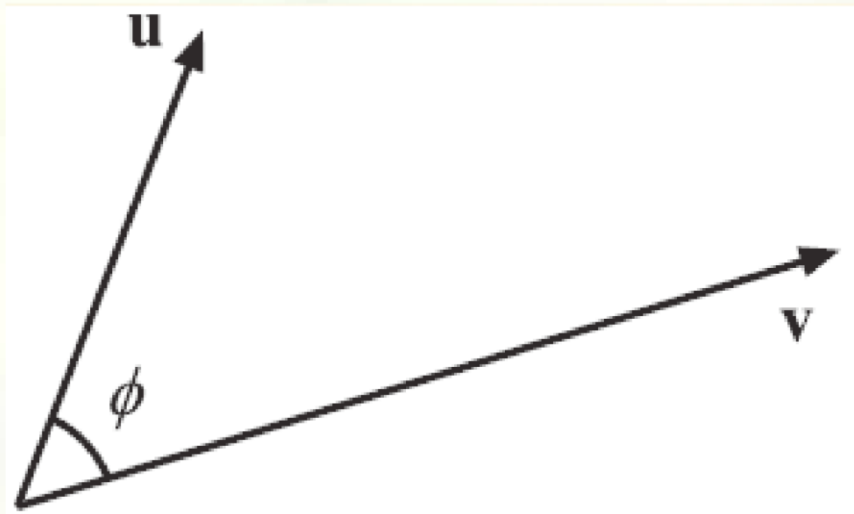
- Or line, or ray, or just linear interpolation

$$\mathbf{p} = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix} = \begin{bmatrix} (1-t)x_0 + t x_1 \\ (1-t)y_0 + t y_1 \\ (1-t)z_0 + t z_1 \end{bmatrix}$$

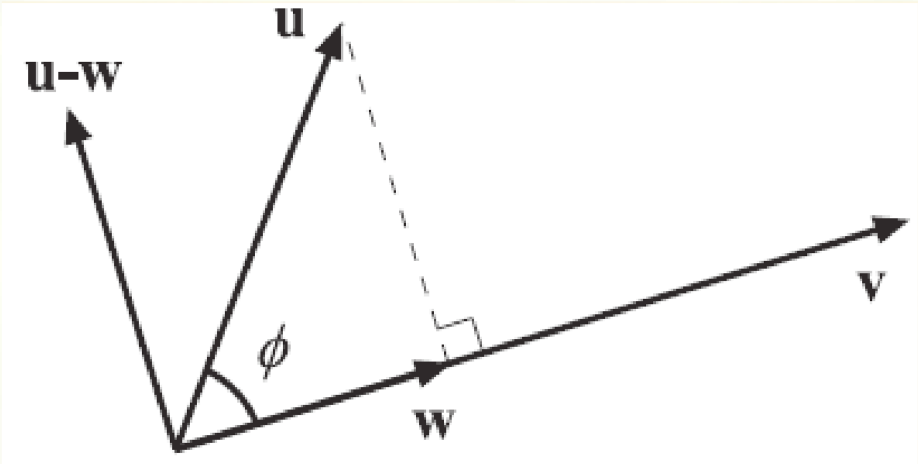
- Line segment $0 \le t \le 1$
- Ray $0 \le t \le \infty$
- Line $-\infty \le t \le \infty$

Vector dot product



$$\mathbf{u} \cdot \mathbf{v} = rx + sy + tz = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\phi)$$

Projection



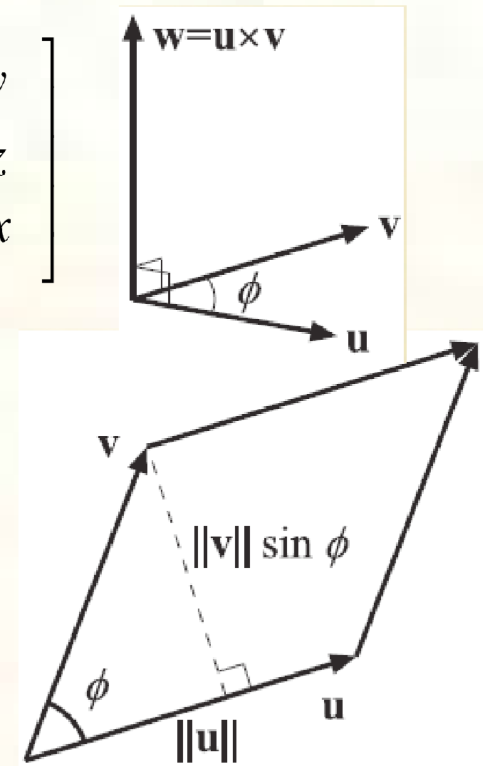
- Projection (**u** component parallel to **v**) $\mathbf{w} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$
- Rejection (**u** component orthogonal to **v**) $\mathbf{u} - \mathbf{w}$
- Particularly useful when vectors are normalized

Cross Product

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{bmatrix} sz - ty \\ tx - rz \\ ry - sx \end{bmatrix}$$

- $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$
- $\|\mathbf{w}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\phi)$
- $\|\mathbf{w}\|$ area of parallelogram
- use right-hand rule
- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

  $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$



Q: What is an application of cross product?

A: Compute the normal direction of a triangle

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei - afh + bfg - bdi + cdh - ceg$$

- $\det(\mathbf{M}^T) = \det(\mathbf{M})$
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
- if $\det(\mathbf{M}) = 0$, **M** is singular, has no inverse

Plane equation

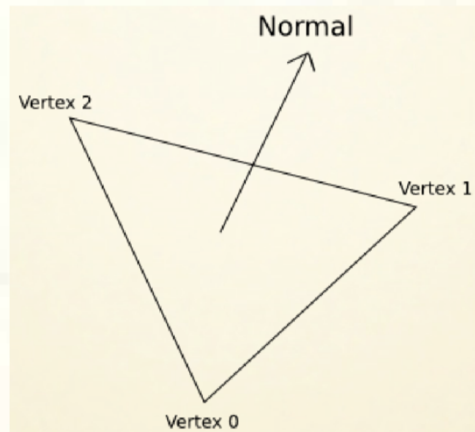
- Given normal vector **N** orthogonal to the plane and any point **p** in the plane $\mathbf{N} \cdot \mathbf{p} + d = 0$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + d = ax + by + cz + d = 0$$

- For a triangle

$$\mathbf{N} = norm((\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0))$$

- Order matters, usually CCW



It is easy to check whether a given point is on one or another side of the plane

Homogeneous coordinates

To represent transformations among affine frames, we can loft the problem up into 3-space, adding a third component to every point:

$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$

$$= \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{u} & \mathbf{w} & \mathbf{t} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= x \cdot \mathbf{u} + y \cdot \mathbf{w} + 1 \cdot \mathbf{t}$$

Note that $[a\ c\ 0]^T$ and $[b\ d\ 0]^T$ represent vectors and $[t_x\ t_y\ 1]^T$, $[x\ y\ 1]^T$ and $[x'\ y'\ 1]^T$ represent points.

Homogeneous coordinates

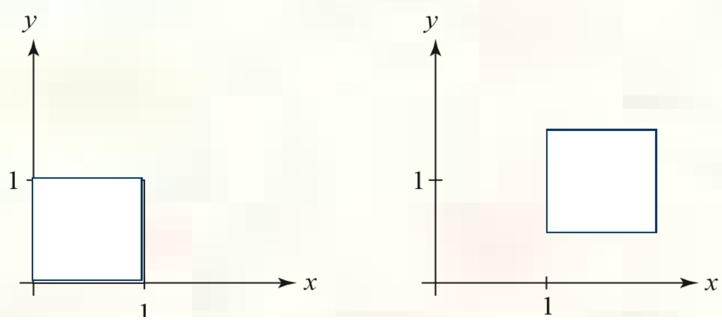
This allows us to perform translation as well as the linear transformations as a matrix operation:

$$\mathbf{p}' = \mathbf{M_T p}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + t_x$$

$$y' = y + t_y$$

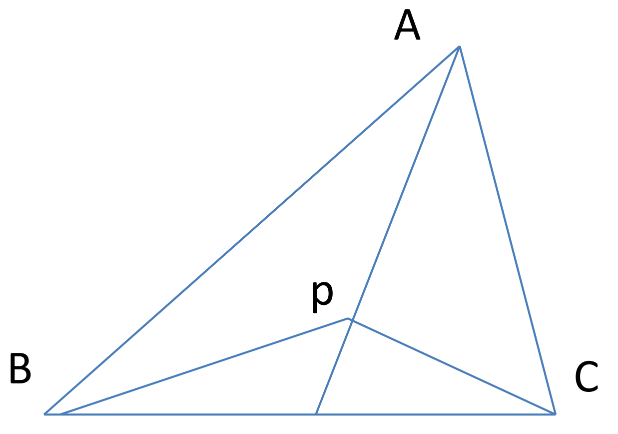


$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Barycentric coords from area ratios

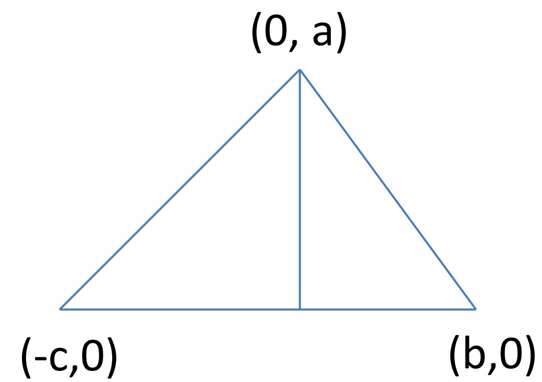
- A geometric interpretation of Barycentric coordinates is through the area ratios

$$\alpha = \frac{\mathrm{SArea}(\mathbf{p}BC)}{\mathrm{SArea}(ABC)} \quad \beta = \frac{\mathrm{SArea}(A\mathbf{p}C)}{\mathrm{SArea}(ABC)} \quad \gamma = \frac{\mathrm{SArea}(AB\mathbf{p})}{\mathrm{SArea}(ABC)}$$



Area

$$2\text{xArea}(ABC) = \left\|\begin{matrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{matrix}\right\|$$



Invariant under translation, rotation

Barycentric coords from area ratios

$$\alpha = \frac{\left\|\begin{matrix} \mathbf{p_x} & B_x & C_x \\ \mathbf{p_y} & B_y & C_y \\ 1 & 1 & 1 \end{matrix}\right\|}{\left\|\begin{matrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{matrix}\right\|} \quad \beta = \frac{\left\|\begin{matrix} A_x & \mathbf{p_x} & C_x \\ A_y & \mathbf{p_y} & C_y \\ 1 & 1 & 1 \end{matrix}\right\|}{\left\|\begin{matrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{matrix}\right\|} \quad \gamma = \frac{\left\|\begin{matrix} A_x & B_x & \mathbf{p_x} \\ A_y & B_y & \mathbf{p_y} \\ 1 & 1 & 1 \end{matrix}\right\|}{\left\|\begin{matrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{matrix}\right\|}$$

Affine and convex combinations

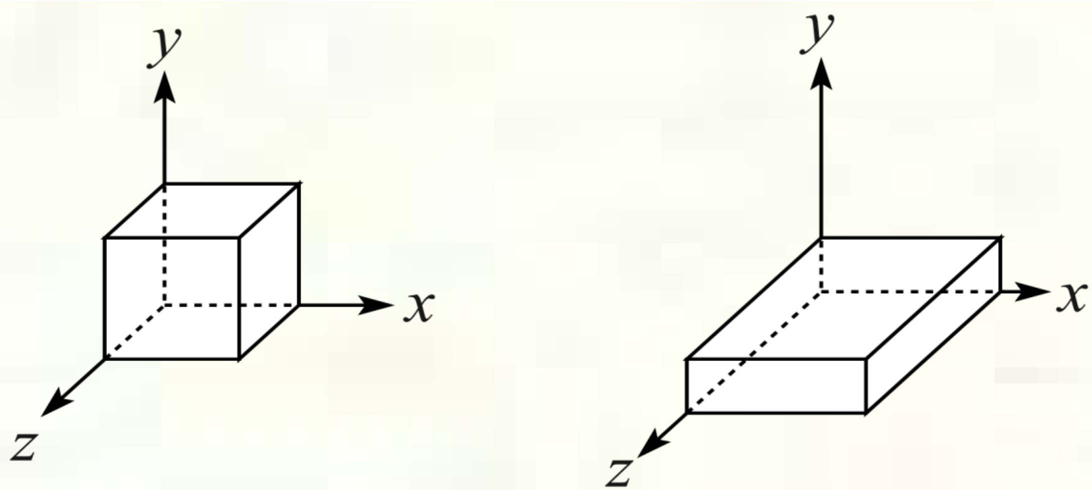
- Note that we seem to have added points together, which we said was illegal, but as long as they have coefficients that sum to one, it is ok. Why?
- We call this an affine combination. More generally

$$\mathbf{p} = \alpha_1 \mathbf{p_1} + \ldots + \alpha_n \mathbf{p_n} \qquad \sum_{i=1}^{n} \alpha_i = 1$$

- If all the coefficients are positive, we call this a convex combination

Basic 3-D transformations: scaling

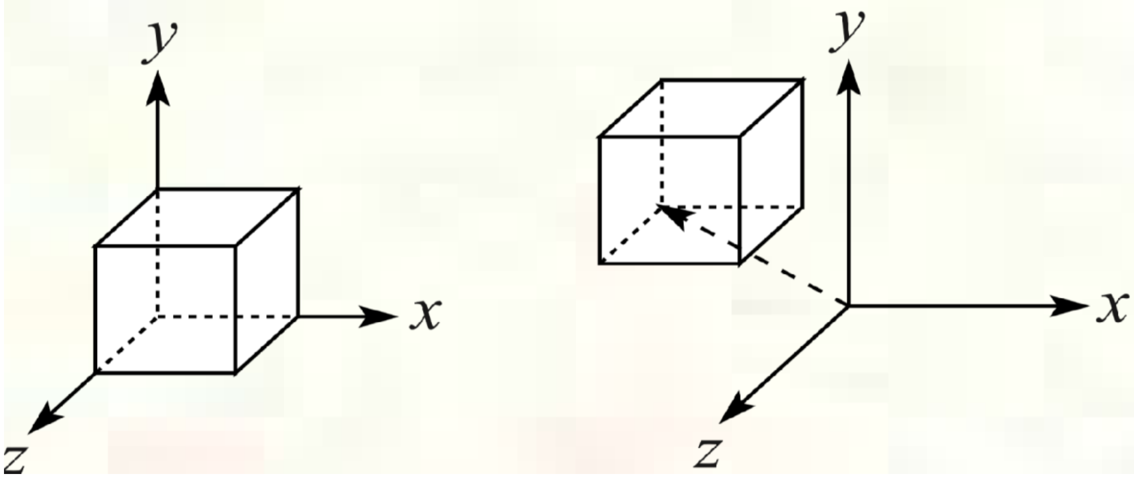
- Some of the 3-D transformations are just like the 2-D ones
- For example, scaling:



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



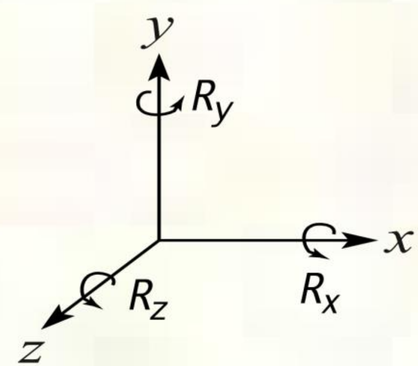
Rotation in 3D

- Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

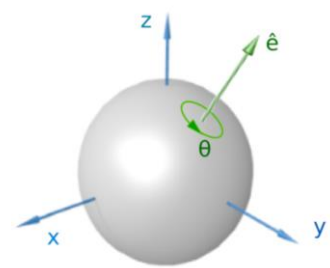
$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

Rotation in 3D

- Rotation is also more complicated in 3D
- Two rotations generally do not communicate
  - Rotation along z followed by Rotation along x is different from Rotation along x first followed by Rotation along z
- Quaternion



$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

A quaternion rotation $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ (with $\mathbf{q} = q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$) can be algebraically manipulated into a matrix rotation $\mathbf{p}' = \mathbf{R}\mathbf{p}$, where R is the rotation matrix given by[4]:

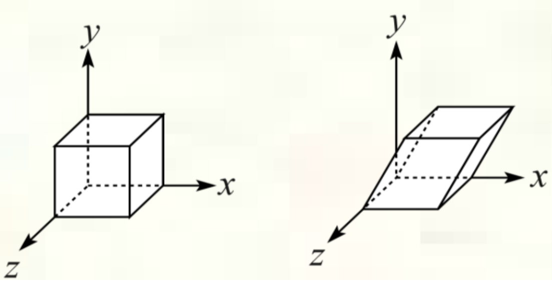
$$\mathbf{R} = \begin{bmatrix} 1-2s(q_j^2+q_k^2) & 2s(q_iq_j-q_kq_r) & 2s(q_iq_k+q_jq_r) \\ 2s(q_iq_j+q_kq_r) & 1-2s(q_i^2+q_k^2) & 2s(q_jq_k-q_iq_r) \\ 2s(q_iq_k-q_jq_r) & 2s(q_jq_k+q_iq_r) & 1-2s(q_i^2+q_j^2) \end{bmatrix}$$

Here $s = ||q||^{-2}$ and if q is a unit quaternion, $s = 1$.

Shearing in 3D

- Shearing is also more complicated. Here is on example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



- We call this a shear with respect to the x-z plane

Preservation of affine combinations

- A transformation *F* is an affine transformation if it preserves affine combinations:

$$F(\alpha_1 \mathbf{p_1} + \ldots + \alpha_n \mathbf{p_n}) = \alpha_1 F(\mathbf{p_1}) + \ldots + \alpha_n F(\mathbf{p_n}) \qquad \sum_{i=1}^{n} \alpha_i = 1$$

- One special example is a matrix that drops a dimension. For example:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- This transformation, known as an orthographic projection, is an affine transformation. We'll use this fact later…

Properties of affine transformations

- Here are some useful properties of affine transformations:
  - Lines map to lines
  - Parallel lines remain parallel
  - Midpoints map to midpoints (in fact, ratios are always preserved)

Next Lecture

- More about ray tracing, math, and transforms

- Special thanks for Don Fussell for many of the slides

Questions?