Discrete Differential Geometry

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Motivation

- Smoothness
  - Mesh smoothing

- Curvature
  - Adaptive simplification

- Parameterization
Motivation

• Triangle shape
  ➡️ Remeshing

• Principal directions
  ➡️ Quad remeshing
Differential Geometry


Leonard Euler (1707 - 1783)  
Carl Friedrich Gauss (1777 - 1855)
Parametric Curves

\[
x = x(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
\]

\[
\dot{x}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \neq 0
\]

\[t \in [a, b] \subset \mathbb{R}\]

“velocity” of particle on trajectory
Parametric Curves

\[ \alpha_1(t) = (\cos(t), \sin(t)) \]

\[ \alpha_2(t) = (\cos(2t), \sin(2t)) \]

Same direction, different speed
Length of a Curve

- Chord length
  \[ S = \sum_i ||\Delta x_i|| = \sum_i ||\frac{\Delta x_i}{\Delta t}|| \Delta t \]
  \[ \Delta x_i = x_{i+1} - x_i \]

- Arc length
  \[ s = s(t) = \int_a^t ||\dot{x}|| \, dt \]
Examples

$\alpha(t) = (a \cos(t), a \sin(t)), \ t \in [0,2\pi]$

$\alpha'(t) = (-a \sin(t), a \cos(t))$

$L(\alpha) = \int_{0}^{2\pi} |\alpha'(t)| \, dt$

$= \int_{0}^{2\pi} \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)} \, dt$

$= a \int_{0}^{2\pi} \, dt = 2\pi a$

Many possible parameterizations

Length of the curve does not depend on parameterization!
Arc Length Parameterization

- Re-parameterization $\mathbf{x}(u(t))$

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{dx}{du} \frac{du}{dt} = \mathbf{x}(u(t)) \dot{u}(t)$$

- Arc length parameterization

$$s = s(t) = \int_a^t \|\dot{\mathbf{x}}\| \, dt$$

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- parameter value $s$ for $\mathbf{x}(s)$ equals length of curve from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\|\dot{\mathbf{x}}(s)\| = 1 \rightarrow \dot{\mathbf{x}}(s) \cdot \ddot{\mathbf{x}}(s) = 0$$
Curvature

$x(t)$ a curve parameterized by arc length

The *curvature* of $x$ at $t$: $\kappa = \|\ddot{x}(t)\|

$\dot{x}(t)$ — the tangent vector at $t$

$\ddot{x}(t)$ — the *change* in the tangent vector at $t$

$R(t) = 1/\kappa(t)$ is the *radius of curvature* at $t$
Examples

**Straight line**

\[ \alpha(s) = us + v, \ u,v \in \mathbb{R}^2 \]

\[ \alpha'(s) = u \]

\[ \alpha''(s) = 0 \quad \Rightarrow \quad |\alpha''(s)| = 0 \]

**Circle**

\[ \alpha(s) = (a \cos(s/a), a \sin(s/a)), \ s \in [0,2\pi a] \]

\[ \alpha'(s) = (-\sin(s/a), \cos(s/a)) \]

\[ \alpha''(s) = (-\cos(s/a)/a, -\sin(s/a)/a) \quad \Rightarrow \quad |\alpha''(s)| = 1/a \]
The Normal Vector

\[ \mathbf{x}'(t) = \mathbf{T}(t) \] - tangent vector

\[ |\mathbf{x}'(t)| \] - velocity

\[ \mathbf{x}''(t) = \mathbf{T}'(t) \] - normal direction

\[ |\mathbf{x}''(t)| \] - curvature

If \( |\mathbf{x}''(t)| \neq 0 \), define \( \mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)| \)

Then \( \mathbf{x}''(t) = \mathbf{T}'(t) = \kappa(t)\mathbf{N}(t) \).
The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the osculating plane at $s$. 
The Binormal Vector

For points $s$, s.t. $\kappa(s) \neq 0$, the **binormal vector** $B(s)$ is defined as:

$$B(s) = T(s) \times N(s)$$

The binormal vector defines the osculating plane
The Frenet Frame

\[ T = \frac{\dot{x}}{\|\dot{x}\|} \]
\[ N = \frac{\ddot{x}}{\|\ddot{x}\|} \]
\[ B = T \times N \]

tangent

normal

binormal
The Frenet Frame

\{ \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s) \} \text{ form an orthonormal basis for } \mathbb{R}^3 \text{ called the Frenet frame}

How does the frame change when the particle moves?

What are \mathbf{T}'(s), \mathbf{N}'(s), \mathbf{B}'(s) in terms of \dot{\mathbf{T}}, \dot{\mathbf{N}}, \dot{\mathbf{B}}?
The Frenet Frame

- Frenet-Serret formulas
  \[
  \dot{T} = \kappa N + \kappa N \\
  \dot{N} = -\kappa T + \tau B \\
  \dot{B} = -\tau N
  \]
- curvature \( \kappa = \|\ddot{x}\| \)
- torsion \( \tau = \frac{1}{\kappa^2} \det [\dot{x}, \ddot{x}, \dddot{x}] \)

(arc-length parameterization)
Curvature and Torsion

- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Invariant under rigid (translation+rotation) motion
- Define curve uniquely up to rigid motion
Surfaces
Differential Geometry: Surfaces

\[ \mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2 \]
Differential Geometry: Surfaces

- Continuous surface

\[ x(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} , \ (u, v) \in \mathbb{R}^2 \]

- Normal vector

\[ n = \frac{(x_u \times x_v)}{\|x_u \times x_v\|} \]

- assuming regular parameterization, i.e.

\[ x_u \times x_v \neq 0 \]
Normal Curvature

\[ \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \]

\[ \mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} \]
Surface Curvature

- Principal Curvatures
  - maximum curvature \( \kappa_1 = \max_{\phi} \kappa_n(\phi) \)
  - minimum curvature \( \kappa_2 = \min_{\phi} \kappa_n(\phi) \)

- Mean Curvature
  \[ H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi \]

- Gaussian Curvature
  \[ K = \kappa_1 \cdot \kappa_2 \]
Principal Curvature

**Euler’s Theorem:** Planes of principal curvature are **orthogonal** and independent of parameterization.

\[ \kappa(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta \quad \theta = \text{angle with } \kappa_1 \]
Curvature

\[ \kappa_1 = \max_{\phi} \kappa_{\eta}(\phi) \]

\[ \kappa_2 = \min_{\phi} \kappa_{\eta}(\phi) \]

\[ H = \frac{1}{2}(\kappa_1 + \kappa_2) \]

\[ K = \kappa_1 \cdot \kappa_2 \]
Surface Classification

**Isotropic**

Equal in all directions

- spherical
- planar

**Anisotropic**

Distinct principal directions

- elliptic $k_1 > 0$, $k_2 > 0$
- parabolic $k_1 > 0$, $k_2 = 0$
- hyperbolic $k_1 > 0$, $k_2 < 0$

Developable

$K = 0$
Principal Directions
Gauss-Bonnet Theorem

For ANY closed manifold surface with Euler number $\chi = 2 - 2g$:

$$ \int K = 2\pi \chi $$

$$ \int K(\text{dolphin}) = \int K(\text{cow}) = \int K(\text{sphere}) = 4\pi $$
Gauss-Bonnet Theorem Example

- **Sphere**
  - $k_1 = k_2 = \frac{1}{r}$
  - $K = k_1 k_2 = \frac{1}{r^2}$
  - $\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$

- **Manipulate sphere**
  - New **positive** + **negative** curvature
  - Cancel out!
Fundamental Forms

• First fundamental form

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} x_u^T x_u & x_u^T x_v \\ x_u^T x_u & x_v^T x_v \end{bmatrix}$$

• Second fundamental form

$$II = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} x_{uu}^T n & x_{uv}^T n \\ x_{uv}^T n & x_{vv}^T n \end{bmatrix}$$
Fundamental Forms

- **I** and **II** allow to measure
  - length, angles, area, curvature
  - arc element
    
    $$ds^2 = Edu^2 + 2F dudv + G dv^2$$

- area element
  
  $$dA = \sqrt{EG - F'^2} dudv$$
Fundamental Forms

- Normal curvature = curvature of the normal curve at point $c \in x(u, v)$, $p \in c$

- Can be expressed in terms of fundamental forms as

$$
\kappa_n(t) = \frac{t^T II t}{t^T I t} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}
$$

$$
\begin{align*}
t &= ax_u + bx_v \\
\bar{t} &= \begin{pmatrix} a \\ b \end{pmatrix}
\end{align*}
$$
Intrinsic Geometry

• Properties of the surface that only depend on the first fundamental form
  – length
  – angles
  – Gaussian curvature

\[ K = \frac{ \begin{vmatrix} \frac{1}{2} E_u & F_u - \frac{1}{2} E_v \\ E & F \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{2} E_v & \frac{1}{2} G_u \\ E & F \end{vmatrix} - \frac{1}{2} E_v E G}{(E G - F^2)^2} \]
Laplace Operator $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ \hspace{1cm} $\Delta f : \mathbb{R}^3 \rightarrow \mathbb{R}$

\[ \Delta f = \nabla \cdot \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \in \mathbb{R} \]

- Laplace operator
- gradient operator
- 2nd partial derivatives
- scalar function in Euclidean space
- divergence operator
- Cartesian coordinates

\[ \text{grad} \ f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

\[ \text{div} \ F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]
Laplace-Beltrami Operator

- Extension to functions on manifolds

\[ \Delta_S f = \text{div}_S \nabla_S f \in \mathbb{R} \]
Laplace-Beltrami Operator

- For coordinate function(s)

\[ f(x, y, z) = x \]

\[ \Delta_s x = \text{div}_s \nabla_s x = -2Hn \quad \in \mathbb{R}^3 \]
Discrete Differential Operators

- **Assumption:** Meshes are piecewise linear approximations of smooth surfaces

- **Approach:** Approximate differential properties at point $v$ as finite differences over local mesh neighborhood $N(v)$
  - $v = $ mesh vertex
  - $N_d(v) = d$-ring neighborhood

- **Disclaimer:** many possible discretizations, none is “perfect”
Functions on Meshes

- Function $f$ given at mesh vertices $f(v_i) = f(x_i) = f_i$
- Linear interpolation to triangle $x \in \{x_i, x_j, x_k\}$

$$f(x) = f_i B_i(x) + f_j B_j(x) + f_k B_k(x)$$
Gradient of a Function

\[ f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x}) \]

\[ \nabla f(\mathbf{x}) = f_i \nabla B_i(\mathbf{x}) + f_j \nabla B_j(\mathbf{x}) + f_k \nabla B_k(\mathbf{x}) \]

Steepest ascent direction perpendicular to opposite edge

\[ \nabla B_i(\mathbf{x}) = \nabla B_i = \frac{\left( \mathbf{x}_k - \mathbf{x}_j \right)^\perp}{2A_T} \]

Constant in the triangle
Gradient of a Function

\[ B_i(x) + B_j(x) + B_k(x) = 1 \]

\[ \nabla B_i + \nabla B_j + \nabla B_k = 0 \]

\[ \nabla f(x) = (f_j - f_i) \nabla B_j(x) + (f_k - f_i) \nabla B_k(x) \]

\[ \nabla f(x) = (f_j - f_i) \frac{(x_i - x_k)}{2A_T} + (f_k - f_i) \frac{(x_j - x_i)}{2A_T} \]
Discrete Laplace-Beltrami First Approach

• Laplace operator: \( \Delta f = \text{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \)

• In 2D: \( \Delta f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)

• On a grid – finite differences discretization:

\[
\Delta f(x_i, y_i) = \frac{f(x_{i+1}, y_i) - f(x_i, y_i)}{h^2} + \frac{f(x_{i-1}, y_i) - f(x_i, y_i)}{h^2} + \\
\frac{f(x_i, y_{i+1}) - f(x_i, y_i)}{h^2} + \frac{f(x_i, y_{i-1}) - f(x_i, y_i)}{h^2}
\]
Discrete Laplace-Beltrami Uniform Discretization

\[ \Delta f = \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) = \]

\[ = |N_1(v)| f(v) - \sum_{v_i \in N_1(v)} f(v_i) \]

Normalized:

\[ \Delta f = \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) = \]

\[ = f(v) - \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} f(v_i) \]
Discrete Laplace-Beltrami—Second Approach

• Laplace-Beltrami operator: \( \Delta_s f = \text{div}_s \nabla_s f \)

• Compute integral around vertex

\[
\int_{A(v)} \Delta f(u) dA
\]
Discrete Laplace-Beltrami Cotangent Formula

Plugging in expression for gradients gives:

$$
\Delta f(v) = \sum_{v_i \in N_1(v)} w_i (f(v_i) - f(v))
$$

$$
= \frac{1}{2A(v)} \sum_{v_i \in N_1(v)} (\cot \alpha_i + \cot \beta_i) (f(v_i) - f(v))
$$
Discrete Normal

$$\Delta_s x = \text{div}_s \nabla_s x = -2Hn$$

$$n(x) = \sum_{v_i \in N_1(v)} w_i (x_i - x) \quad \sum_i w_i = 1$$

$$= \left( \sum_{x_i \in N_1(x)} w_i x_i \right) - x$$

$$\sum w_i x_i$$
Discrete Curvatures

- Mean curvature
  \[ H = \| \Delta s x \| \]

- Gaussian curvature
  \[ G = \frac{(2\pi - \sum_j \theta_j)}{A} \]

- Principal curvatures
  \[ \kappa_1 = H + \sqrt{H^2 - G} \]
  \[ \kappa_2 = H - \sqrt{H^2 - G} \]
Example: Mean Curvature
Example: Gaussian Curvature

Discrete Gauss-Bonnet (Descartes) theorem:
\[
\sum_v K_v = \sum_v \left[ 2\pi - \sum_i \theta_i \right] = 2\pi \chi
\]