# CS 336: Solutions for Homework 1

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September 12, 2009

Note: Many of these problems have multiple correct answers. Total possible point value was 57. Homework scores are normalized to out of 20 points.

2. [2 pts] $\{x \in \mathbb{Q} \ : \ x > 0\}$
3. [2 pts] $\{x \in \mathbb{R} \setminus \mathbb{Q} \ : \ x > 0\}$
4. [2 pts] $\{x \in \mathbb{R} \ : \ x > 0 \wedge x \notin 3\mathbb{Z}\}$
5. [2 pts] $\{f : \mathbb{R} \to \mathbb{Z}\}$
6. [3 pts] A function $f : A \to B$ is 1-to-1 if each point in A is sent by $f$ to a different point in B. The usual way of expressing this in logic is:

$$\forall x, y \in A \ : \ f(x) = f(y) \ \Rightarrow \ x = y$$

which says that if f sends x and y to the same point in B, then x and y are actually the same. Another valid variant of this is

$$\forall x, y \in A \ : \ x \neq y \Rightarrow \ f(x) \neq f(y)$$

This is the contrapositive of the first statement, so the two are logically equivalant.

7. [3 pts] A function $f : A \to B$ is onto if $f$ sends something to each point in B.

$$\forall y \in B \ \exists x \in A \ s.t. \ f(x) = y$$

8. [12 pts]

| $A$ | $B$ | $\neg A$ | $A \Rightarrow B$ | $(A \Rightarrow B) \ \wedge \ (\neg A)$ |
|---|---|---|---|---|
| F | F | T | T | T |
| F | T | T | T | T |
| T | F | F | F | F |
| T | T | F | T | F |

| $A$ | $B$ | $\neg A$ | $B \Rightarrow \neg A$ | $A \wedge (B \Rightarrow \neg A)$ | $\neg B$ | $A \wedge (B \Rightarrow \neg A) \wedge \neg B$ |
|---|---|---|---|---|---|---|
| F | F | T | T | F | T | F |
| F | T | T | T | F | F | F |
| T | F | F | T | T | T | T |
| T | T | F | F | F | F | F |

| $A$ | $B$ | $\neg A$ | $\neg B$ | $\neg B \Rightarrow \neg A$ | $A \wedge (\neg B \Rightarrow \neg A)$ | $A \wedge (\neg B \Rightarrow \neg A) \wedge \neg B$ |
|---|---|---|---|---|---|---|
| F | F | T | T | T | F | F |
| F | T | T | F | T | F | F |
| T | F | F | T | F | F | F |
| T | T | F | F | T | T | F |

9. (a) [6 pts] The first step to this question is understanding what these logical statements mean. Statement 1 says that for every x, there exists a y larger than x. In plain english this means there is no largest element in the set. Statement 2 says there exists an x which is smaller than all other elements, or in plain english that the set has a minimum.

| | statement 1 | statement 2 |
|---|---|---|
| 1 | true | true |
| 2 | false | false |
| 3 | false | true |
| 4 | false | true |
| 5 | true | false |
| 6 | true | false |

Look closely at the formal logic statements with regard to $S = \emptyset$ (number 6). Statement 1 is vacuously true: S has no elements, so $\forall x \in S : P$ is true for any statement P. On the other hand, statement 2 is *not* vacuously true. The statement requires an element of S, which cannot hold because S is empty.

(b) [4 pts] 1: $\exists x \in S \; s.t. \; \forall y \in S \; x \geq y$
2: $\forall x \in S \; \exists y \in S \setminus \{x\} \; s.t. \; x \geq y$

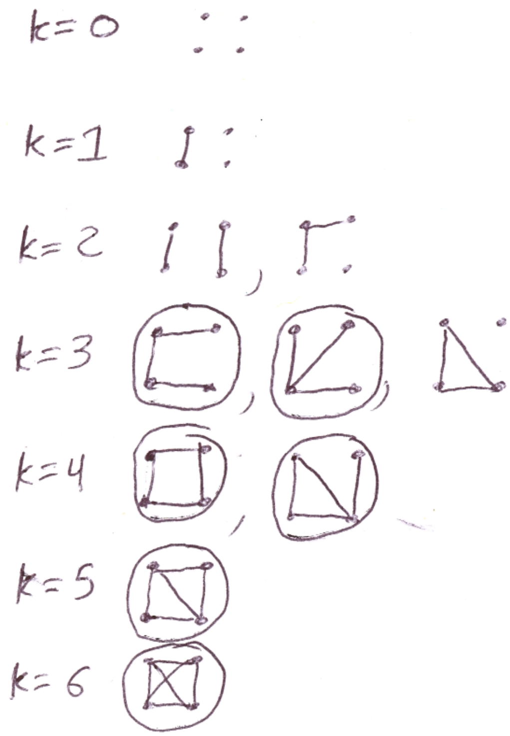
10. [5 pts] Input: Graph G=(V,E), where V is the set of all UT students and $(v_1, v_2) \in E$ iff student $v_1$ dislikes student $v_2$.
Output: Set $O = \{\kappa_1, \kappa_2, \ldots\}$ such that:

(a) Each $\kappa_i$ is a set of UT students ($\forall \kappa_i \in O : \kappa_i \subseteq V$)
(b) Each student is in exactly one $\kappa_i$ ($\forall s \in V \; \exists! \, \kappa_i \in O \; s.t. \; s \in g_i$).
(c) Within any $\kappa_i$, no members of the group dislike each other ($\forall s_1, s_2 \in \kappa_i : (s_1, s_2) \notin E$).
(d) O is a minimal size set satisfying the above conditions.

In mathematics, O is called a *partition* of V.

11. [6 pts] A graph with 4 nodes can have up to $\frac{4 \cdot (4-1)}{2} = 6$ edges. Graphs with different numbers of edges are necessarily different graphs, so we can subdivide this problem into the task of enumerating all graphs with 4 nodes and k edges.

In total, there are six connected graphs (circled below).




12\. - 15. [Not graded] See instructor for solutions.

16\. [5 pts] Each function is defined by where it sends exactly two elements.

| | $f_i(1)$ | $f_i(2)$ |
|---|---|---|
| $f_1$ | 1 | 1 |
| $f_2$ | 1 | 2 |
| $f_3$ | 2 | 1 |
| $f_4$ | 2 | 2 |

Two functions are 1-1 ($f_2$ and $f_3$). Two functions are onto ($f_2$ and $f_3$).

17\. [5 pts] Our induction hypothesis is: "For a given n: $1 + 2 + \ldots + n = n \cdot (n + 1)/2$."

Base case: let $n = 1$. $1 = 1 \cdot (1 + 1)/2$. So the base case holds.
Induction step: Take $k \geq 1$. By induction hypothesis, we assume the statement holds true for $n = k$, and we prove for $n = k + 1$. By I.H., $1 + 2 + \ldots + k = k \cdot (k + 1)/2$. Adding $k + 1$ to both sides of the equation yields:

$$\begin{aligned} 1 + 2 + \ldots + k + k + 1 &= k \cdot (k + 1)/2 + (k + 1) \\ &= (k + 1) \cdot (k + 2)/2 \end{aligned}$$

as desired.