CS 336: Solutions for Homework 2

Isaac Levy

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Total possible points: 34. Homework scores are normalized to out of 20 points.

1. [2 pts each] Problems from Sept 8th lecture notes, page 11.
   (a) $\exists v_1, v_2 \in V$ s.t. $v_1 \neq v_2 \ \wedge \ (v_1, v_2) \in E$
   (b) $\exists v_1, v_2 \in V$ s.t. $v_1 \neq v_2 \ \wedge \ (v_1, v_2) \notin E$
   (c) $\exists A \subseteq V$ with $|A| = B$ s.t. $\forall v_1, v_2 \in A \ [v_1 \neq v_2 \ \Rightarrow \ (v_1, v_2) \in E]$
   (d) $\exists A \subseteq V$ with $|A| = B$ s.t. $\forall v_1, v_2 \in A \ [v_1 \neq v_2 \ \Rightarrow \ (v_1, v_2) \notin E]$
   (e) $\exists f : V \to \{1, 2, 3\}$ s.t. $\forall v_1, v_2 \in V \ [v_1 \neq v_2 \ \wedge \ (v_1, v_2) \in E] \ \Rightarrow f(v_1) \neq f(v_2)$
   (f) $\exists A, B \subseteq V$ with $B = V \setminus A$ s.t. $\forall v_1, v_2 \in A \ \ (v_1, v_2) \notin E$ and $\forall w_1, w_2 \in B \ \ (w_1, w_2) \notin E$.
   (g) $\exists A, B \subseteq V$ with $B = V \setminus A$ s.t. $\forall v_1 \in A, v_2 \in B \ \ (v_1, v_2) \notin E$.

2. (a) [1 pt] $A_2 = (1, 2)$
   (b) [1 pt] $A_5 = (1, 5) \setminus \{2, 3, 4\}$
   (c) [2 pts closed form, 6 pts induction] $A_n = (1, n) \setminus \{2, 3, \ldots, n-1\}$

   Proof. We must show that both definitions of A_n are equivalent. Let $S_n = (1, n) \setminus \{2, 3, \ldots, n-1\}$ and let A_n be defined recursively as in the problem. Our induction hypothesis is that for some $n = k$, $A_n = S_n$. We will use induction to prove this statement for all n.

   Base Case. n=1. $A_1 = \emptyset$ by definition. $S_1 = (1, 1) \setminus \emptyset = \emptyset$. So $A_1 = S_1$ and our base case holds.

   Induction. Assume the statement for $n = k$ for some $k \geq 1$ and show it is true for $n = k + 1$.

$$
\begin{aligned}
A_{k+1} &= A_k \cup (k, k+1) && \text{by def of A since } k+1 \geq 2 \\
&= S_k \cup (k, k+1) && \text{by induction hypothesis} \\
&= ((1, k) \setminus \{2, 3, \ldots, k-1\}) \cup (k, k+1) && \text{by definition of S} \\
&= ((1, k) \cup (k, k+1)) \setminus \{2, 3, \ldots, k-1\} && \text{b/c } (k, k+1) \cap \{2, \ldots, k-1\} = \emptyset \\
&= (1, k+1) \setminus \{2, 3, \ldots, k-1, k\} && \text{arithmetic} \\
&= S_{k+1} && \text{by definition of S}
\end{aligned}
$$

   □

   (d) [2 pts] One solution is: $\displaystyle\bigcup_{n \in \mathbb{N}} A_n = (1, \infty) \setminus \mathbb{N}$

3. (a) [1 pt] $B_2 = \emptyset \cup [1, 2) = [1, 2)$

(b) [1 pt] $B_5 = [1, 5)$

(c) [2 pts closed form, 6 pts induction] $B_n = [1, n)$

Proof. We must show that both definitions of B_n are equivalent. Let $T_n = [1, n)$ and let B_n be defined recursively as in the problem. Our induction hypothesis is that for some $n = k$, $B_n = T_n$. We will use induction to prove this statement for all n.

Base Case. n=1. $B_1 = \emptyset$ by definition. $S_1 = [1, 1) = \emptyset$. So $B_1 = S_1$ and our base case holds.

Induction. Assume the statement for $n = k$ for some $k \geq 1$ and show it is true for $n = k + 1$.

$$\begin{aligned}
B_{k+1} &= B_k \cup [k, k+1) && \text{by def of B since } k+1 \geq 2 \\
&= T_k \cup [k, k+1) && \text{by induction hypothesis} \\
&= [1, k) \cup [k, k+1) && \text{by definition of T} \\
&= [1, k+1) && \text{arithmetic} \\
&= T_{k+1} && \text{by definition of S}
\end{aligned}$$

$\square$

(d) [2 pts] $\bigcup_{n \in \mathbb{N}} B_n = [1, \infty)$