CS 336: Solutions for Homework 3

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1 (a) [1 pt] $F(1,1) = 1$, because $F(1,1)$ falls into the $F(n,n)$ case.

1 (b) [1 pt] $F(2,1)$ falls into the third case. So we must evaluate recursively:

$$F(2,1) = F(1,1) + F(1,0) = 1 + 1 = 2$$

1 (c) [8 pts] I like to start these proofs by defining a second function that explicitly encapsulates the closed form. Then I try to prove that the two functions are equal. This is a personal preference and is definitely not required in your homeworks. If you find yourself getting confused about which variables are fixed and what can be assumed, try out this form to make the I.H. simpler. Anyway, back to the proof... First we define the closed form:

$$H(n,k) = \frac{n!}{(n-k)! \cdot k!} \text{ for } n \geq k \geq 0$$

There are several induction statements that can yield correct induction proofs. I will provide proofs for two possible statements.

Note: $\mathbb{N}^0$ is the set of all integers greater than or equal to 0. That is, it is $\mathbb{N} \cup \{0\}$.

i. Induction Statement:

$$\forall c \in \mathbb{N}^0 : \forall n, k \in \mathbb{N}^0 : [(n \geq k \wedge n + k = c) \implies F(n,k) = H(n,k)]$$

We will induct on c. This induction requires strong induction (*why?*).

Base case. c=0. Then the only n,k which can satisfy the antecedent of the hypothesis is if $n = k = 0$.

By definition, $F(0,0) = 1$. Also $H(0,0) = \frac{1}{1 \cdot 1} = 1$. So the base case holds.

Induction. Fix $l \in \mathbb{N}^0$. Strong Induction Hypothesis: For all $c \leq l$ we assume the statement holds. We prove the statement for $c' = l + 1$.

Take some $n', k' \in \mathbb{N}^0$ s.t. $n' + k' = c'$ and $n' \geq k'$.

First, suppose $n' = k'$ or $k' = 0$. Then by definition of F, $F(n', k') = 1$. By evaluating the closed form on these two cases, we find that $H(n', k') = 1$ as well. Hence if $n' = k'$ or $k' = 0$, then $F(n', k') = H(n', k')$.

Otherwise $n' > k' > 0$. By the recursive case of F, $F(n', k') = F(n' - 1, k') + F(n' - 1, k' - 1)$. Our inductive defintion applies, because $(n' - 1) + (k) = c' - 1 = l$ and $(n' - 1) + (k' - 1) = c' - 2 \leq l$. Hence:

$$
\begin{aligned}
F(n',k') &= F(n'-1,k') + F(n'-1,k'-1) && \text{def of F since } n' > k' > 0. \\
&= H(n'-1,k') + H(n'-1,k'-1) && \text{by I.H.} \\
&= \frac{(n'-1)!}{(n'-1-k')! \cdot k'!} + \frac{(n'-1)!}{((n'-1)-(k'-1))! \cdot (k'-1)!} && \text{by def of H} \\
&= \frac{(n'-1)!}{(n'-k'-1)! \cdot k'!} + \frac{(n'-1)!}{(n'-k')! \cdot (k'-1)!} && \text{minor arith} \\
&= \frac{(n'-k')(n'-1)!}{(n'-k')(n'-k'-1)! \cdot k'!} + \frac{(n'-1)! \cdot (k')}{(n'-k')! \cdot (k'-1)! \cdot (k')} && \text{arithmetic} \\
&= \frac{(n'-k')(n'-1)!}{(n'-k')! \cdot k'!} + \frac{(k')(n'-1)!}{(n'-k')! \cdot (k')!} && \text{arithmetic} \\
&= \frac{(n'-k')(n'-1)! + (k')(n'-1)!}{(n'-k')! \cdot k'!} && \text{arithmetic} \\
&= \frac{(n'-k'+(k'))(n'-1)!}{(n'-k')! \cdot k'!} && \text{arithmetic} \\
&= \frac{(n')(n'-1)!}{(n'-k')! \cdot k'!} && \text{arithmetic} \\
&= \frac{(n')!}{(n'-k')! \cdot k'!} && \text{arithmetic} \\
&= H(n',k') && \text{by def of H}
\end{aligned}
$$

The choice of n', k' were arbitrary, so we have proved the statement $\forall n', k' \in \mathbb{N}^0$ with $n'+k' = c'$ and $n' \geq k'$. Hence we have proved the statement for $c' = l+1$, so the induction step is complete.

By strong induction, the statement is true for all c.

ii. Another possible induction statement is:

$$\forall n \in \mathbb{N}^0 : \forall k \in \{x \in \mathbb{Z} : n \geq x \geq 0\} : F(n,k) = H(n,k)$$

This statement can be proved with weak induction on n.

Base Case. $n = 0$. Then $k = 0$. By definition, $F(0,0) = 1$. Also $H(0,0) = \frac{1}{1 \cdot 1} = 1$. So the base case holds.

Induction. Assume the statement holds for some $n \geq 0$. Prove the statement for $n+1$.

Take $k' \in \{x \in \mathbb{Z} : n+1 \geq x \geq 0\}$. As in the previous proof, split into cases based on when $n = k$, $k = 0$, or when $n > k > 0$. In the third case, the induction hypothesis applies to both $F(n,k')$ and $F(n,k'-1)$ because the statement holds for all $k \in \{x \in \mathbb{Z} : n \geq x \geq 0\}$. So this case proceeds similarly to the previous proof. Now assume we have just shown that $F(n+1,k') = H(n+1,k')$ (this is all the factorial arithmetic above).

The choice of k' was arbitrary, so we have proved the statement $\forall k' \in \{x \in \mathbb{Z} : n+1 \geq x \geq 0\}$. Hence we have proved the statement for n+1. The induction step is complete.

By induction, the statement is true for all n.