Mixed Linear Regression with Multiple Components

Kai Zhong [1]   **Prateek Jain** [2]   **Inderjit S. Dhillon** [3]

[1,3] University of Texas at Austin   [2] Microsoft Research India

[1] `zhongkai@ices.utexas.edu`,   [2] `prajain@microsoft.com`

[3] `inderjit@cs.utexas.edu`

Abstract

In this paper, we study the mixed linear regression (MLR) problem, where the goal is to recover multiple underlying linear models from their unlabeled linear measurements. We propose a non-convex objective function which we show is *locally strongly convex* in the neighborhood of the ground truth. We use a tensor method for initialization so that the initial models are in the local strong convexity region. We then employ general convex optimization algorithms to minimize the objective function. To the best of our knowledge, our approach provides first exact recovery guarantees for the MLR problem with $K \geq 2$ components. Moreover, our method has near-optimal computational complexity $\widetilde{O}(Nd)$ as well as near-optimal sample complexity $\widetilde{O}(d)$ for *constant K*. Furthermore, we show that our non-convex formulation can be extended to solving the *subspace clustering* problem as well. In particular, when initialized within a small constant distance to the true subspaces, our method converges to the global optima (and recovers true subspaces) in time *linear* in the number of points. Furthermore, our empirical results indicate that even with random initialization, our approach converges to the global optima in linear time, providing speed-up of up to two orders of magnitude.

1   Introduction

The mixed linear regression (MLR) [7, 9, 29] models each observation as being generated from one of the K unknown linear models; the identity of the generating model for each data point is also unknown. MLR is a popular technique for capturing non-linear measurements while still keeping the models simple and computationally efficient. Several widely-used variants of linear regression, such as piecewise linear regression [14, 28] and locally linear regression [8], can be viewed as special cases of MLR. MLR has also been applied in time-series analysis [6], trajectory clustering [15], health care analysis [11] and phase retrieval [4]. See [27] for more applications.

In general, MLR is NP-hard [29] with the hardness arising due to lack of information about the model labels (model from which a point is generated) as well as the model parameters. However, under certain statistical assumptions, several recent works have provided poly-time algorithms for solving MLR [2, 4, 9, 29]. But most of the existing recovery gurantees are restricted either to mixtures with $K = 2$ components [4, 9, 29] or require $poly(1/\epsilon)$ samples/time to achieve ϵ-approximate solution [7, 24] (analysis of [29] for two components can obtain ϵ approximate solution in $\log(1/\epsilon)$ samples). Hence, solving the MLR problem with $K \geq 2$ mixtures while using near-optimal number of samples and computational time is still an open question.

In this paper, we resolve the above question under standard statistical assumptions for *constant* many mixture components K. To this end, we propose the following smooth objective function as a

surrogate to solve MLR:

$$f(\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_K) := \sum_{i=1}^{n} \Pi_{k=1}^{K}(y_i - \boldsymbol{x}_i^T \boldsymbol{w}_k)^2, \tag{1}$$

where $\{(\boldsymbol{x}_i, y_i) \in \mathbb{R}^{d+1}\}_{i=1,2,\cdots,N}$ are the data points and $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ are the model parameters. The intuition for this objective is that the objective value is zero when $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ is the global optima and y's do not contain any noise. Furthermore, the objective function is smooth and hence less prone to getting stuck in arbitrary saddle points or oscillating between two points. The standard EM [29] algorithm instead makes a "sharp" selection of mixture component and hence the algorithm is more likely to oscillate or get stuck. This intuition is reflected in Figure 1 (d) which shows that with random initialization, EM algorithm routinely gets stuck at poor solutions, while our proposed method based on the above objective still converges to the global optima.

Unfortunately, the above objective function is non-convex and is in general prone to poor saddle points, local minima. However under certain standard assumptions, we show that the objective is locally strongly convex (Theorem 1) in a small basin of attraction near the optimal solution. Moreover, the objective function is smooth. Hence, we can use gradient descent method to achieve linear rate of convergence to the global optima. But, we will need to initialize the optimization algorithm with an iterate which lies in a small ball around the optima. To this end, we modify the tensor method in [2, 7] to obtain a "good" initialization point. Typically, tensor methods require computation of third and higher order moments which leads to significantly worse sample complexity in terms of data dimensionality d. However, for the special case of MLR, we provide a small modification of the standard tensor method that achieves nearly optimal sample and time complexity bounds for constant K (see Theorem 3) . More concretely, our approach requires $\tilde{O}(d(K \log d)^K)$ many samples and requires $\tilde{O}(Nd)$ computational time; note the exponential dependence on K. Also for constant K, the method has nearly optimal sample and time complexity.

Subspace clustering: MLR can be viewed as a special case of subspace clustering (SC), since each regressor-response pair lies in the subspace determined by this pair's model parameters. However, solving MLR using SC approaches is intractable because the dimension of each subspace is only one less than the ambient dimension, which will easily violate the conditions for the recovery guarantees of most methods (see e.g. Table 1 in [23] for the conditions of different methods). Nonetheless, our objective for MLR easily extends to the subspace clustering problem. That is, given data points $\{\boldsymbol{z}_i \in \mathbb{R}^d\}_{i=1,2,\cdots,N}$, the goal is to minimize the following objective w.r.t. K subspaces (each of dimension at most r):

$$\min_{U_k \in \mathbb{O}^{d \times r}, k=1,2,\cdots,K} f(U_1, U_2, \cdots, U_K) = \sum_{i=1}^{N} \Pi_{k=1}^{K} \langle I_d - U_k U_k^T, \boldsymbol{z}_i \boldsymbol{z}_i^T \rangle. \tag{2}$$

U_k denotes the basis spanned by k-th estimated subspace and $\mathbb{O}^{d \times r} \subset \mathbb{R}^{d \times r}$ denotes the set of orthonormal matrices, i.e., $U^T U = I$ if $U \in \mathbb{O}^{d \times r}$. We propose a power-method style algorithm to alternately optimize (2) w.r.t $\{U_k\}_{k=1,2,\cdots,K}$, which takes only $O(rdN)$ time compared with $O(dN^2)$ for the state-of-the-art methods, e.g. [13, 22, 23].

Although EM with power method [4] shares the same computational complexity as ours, there is no convergence guarantee for EM to the best of our knowledge. In contrast, we provide local convergence guarantee for our method. That is, if $N = \tilde{O}(rK^K)$ and if data satisfies certain standard assumptions, then starting from an initial point $\{U_k\}_{k=1,\cdots,K}$ that lies in a small ball of constant radius around the globally optimal solution, our method converges super-linearly to the globally optimal solution. Unfortunately, our existing analyses do not provide global convergence guarantee and we leave it as a topic for future work. Interestingly, our empirical results indicated that even with randomly initialized $\{U_k\}_{k=1,\cdots,K}$, our method is able to recover the true subspace *exactly* using nearly $O(rK)$ samples.

We summarize our contributions below:

(1) MLR: We propose a non-convex continuous objective function for solving the mixed linear regression problem. To the best of our knowledge, our algorithm is the first work that can handle $K \geq 2$ components with global convergence guarantee in the noiseless case (Theorem 4). Our algorithm has near-optimal linear (in d) sample complexity and near-optimal computational complexity; however, our sample complexity dependence on K is exponential.

(2) Subspace Clustering: We extend our objective function to subspace clustering, which can be optimized efficiently in $O(rdN)$ time compared with $O(dN^2)$ for state-of-the-art methods. We also provide a small basin of attraction in which our iterates converge to the global optima at super-linear rate (Theorem 5).

2 Related Work

Mixed Linear Regression:
EM algorithm without careful initialization is only guaranteed to have local convergence [4, 21, 29]. [29] proposed a grid search method for initialization. However, it is limited to the two-component case and seems non-trivial to extend to multiple components. It is known that exact minimization for each step of EM is not scalable due to the $O(d^2N + d^3)$ complexity. Alternatively, we can use EM with gradient update, whose local convergence is also guaranteed by [4] but only in the two-symmetric-component case, i.e., when $\boldsymbol{w}_2 = -\boldsymbol{w}_1$.

Tensor Methods for MLR were studied by [7, 24]. [24] approximated the third-order moment directly from samples with Gaussian distribution, while [7] learned the third-order moment from a low-rank linear regression problem. Tensor methods can obtain the model parameters to any precision ϵ but requires $1/\epsilon^2$ time/samples. Also, tensor methods can handle multiple components but suffer from high sample complexity and high computational complexity. For example, the sample complexity required by [7] and [24] is $O(d^6)$ and $O(d^3)$ respectively. On the other hand, the computational burden mainly comes from the operation on tensor, which costs at least $O(d^3)$ for a very simple tensor evaluation. [7] also suffers from the slow nuclear norm minimization when estimating the second and third order moments. In contrast, we use tensor method only for initialization, i.e., we require ϵ to be a certain constant. Moreover, with a simple trick, we can ensure that the sample and time complexity of our initialization step is only linear in d and N.

Convex Formulation. Another approach to guarantee the recovery of the parameters is to relax the non-convex problem to convex problem. [9] proposed a convex formulation of MLR with two components. The authors provide upper bounds on the recovery errors in the noisy case and show their algorithm is information-theoretically optimal. However, the convex formulation needs to solve a nuclear norm function under linear constraints, which leads to high computational cost. The extension from two components to multiple components for this formulation is also not straightforward.

Subspace Clustering:
Subspace clustering [13, 17, 22, 23] is an important data clustering problem arising in many research areas. The most popular subspace clustering algorithms, such as [13, 17, 23], are based on a two-stage algorithm – first finding a neighborhood for each data point and then clustering the points given the neighborhood. The first stage usually takes at least $O(dN^2)$ time, which is prohibitive when N is large. On the other hand, several methods such as K-subspaces clustering [18], K-SVD [1] and online subspace clustering [25] do have linear time complexity $O(rdN)$ per iteration, however, there are no global or local convergence guarantees. In contrast, we show locally superlinear convergence result for an algorithm with computational complexity $O(rdN)$. Our empirical results indicate that random initialization is also sufficient to get to the global optima; we leave further investigation of such an algorithm for future work.

3 Mixed Linear Regression with Multiple Components

In this paper, we assume the dataset $\{(\boldsymbol{x}_i, y_i) \in \mathbb{R}^{d+1}\}_{i=1,2,\cdots,N}$ is generated by,

$$z_i \sim \text{multinomial}(\boldsymbol{p}), \quad \boldsymbol{x}_i \sim \mathcal{N}(0, I_d), \quad y_i = \boldsymbol{x}_i^T \boldsymbol{w}_{z_i}^*, \tag{3}$$

where $\boldsymbol{p}$ is the proportion of different components satisfying $\boldsymbol{p}^T \mathbf{1} = 1$, $\{\boldsymbol{w}_k^* \in \mathbb{R}^d\}_{k=1,2,\cdots,K}$ are the ground truth parameters. The goal is to recover $\{\boldsymbol{w}_k^*\}_{k=1,2,\cdots,K}$ from the dataset. Our analysis is based on noiseless cases but we illustrate the empirical performance of our algorithm for the noisy cases, where $y_i = \boldsymbol{x}_i^T \boldsymbol{w}_{z_i}^* + e_i$ for some noise e_i (see Figure 1).

Notation. We use $[N]$ to denote the set $\{1, 2, \cdots, N\}$ and $S_k \subset [N]$ to denote the index set of the samples that come from k-th component. Define $p_{\min} := \min_{k \in [K]}\{p_k\}$, $p_{\max} := \max_{k \in [K]}\{p_k\}$. Define $\Delta \boldsymbol{w}_j := \boldsymbol{w}_j - \boldsymbol{w}_j^*$ and $\Delta \boldsymbol{w}_{kj}^* := \boldsymbol{w}_k^* - \boldsymbol{w}_j^*$. Define $\Delta_{\min} := \min_{j \neq k}\{\|\Delta \boldsymbol{w}_{jk}^*\|\}$ and

$\Delta_{\max} := \max_{j \neq k}\{\|\Delta \boldsymbol{w}_{jk}^*\|\}$. We assume $\frac{\Delta_{\min}}{\Delta_{\max}}$ is independent of the dimension d. Define $\boldsymbol{w} := [\boldsymbol{w}_1; \boldsymbol{w}_2; \cdots; \boldsymbol{w}_K] \in \mathbb{R}^{Kd}$. We denote $\boldsymbol{w}^{(t)}$ as the parameters at t-th iteration and $\boldsymbol{w}^{(0)}$ as the initial parameters. For simplicity, we assume there are $p_k N$ samples from the k-th model in any random subset of N samples. We use $\mathbb{E}[\![X]\!]$ to denote the expectation of a random variable X. Let

$T \in \mathbb{R}^{d \times d \times d}$ be a tensor and T_{ijk} be the i, j, k-th entry of T. We say a tensor is supersymmetric if T_{ijk} is invariant under any permutation of i, j, k. We also use the same notation T to denote the multi-array map from three matrices, $A, B, C \in \mathbb{R}^{d \times r}$, to a new tensor: $[T(A, B, C)]_{i,j,k} = \sum_{p,q,l} T_{pql} A_{pi} B_{qj} C_{lk}$. We say a tensor T is rank-one if $T = \boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}$, where $T_{ijk} = a_i b_j c_k$. We use $\|A\|$ denote the spectral norm of the matrix A and $\sigma_i(A)$ to denote the i-th largest singular value of A. For tensors, we use $\|T\|_{op}$ to denote the operator norm for a supersymmetric tensor T, $\|T\|_{op} := \max_{\|\boldsymbol{a}\|=1} |T(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a})|$. We use $T_{(1)} \in \mathbb{R}^{d \times d^2}$ to denote the matricizing of T in the first order, i.e., $[T_{(1)}]_{i,(j-1)d+k} = T_{ijk}$. Throughout the paper, we use $\widetilde{O}(d)$ to denote $O(d \times \text{polylog}(d))$. We assume K is a constant in general. However, if some numbers depend on K^K, we will explicitly present it in the big O notation. For simplicity, we just include higher-order terms of K and ignore lower-order terms, e.g., $O((2K)^{2K})$ may be replaced by $O(K^K)$.

3.1 Local Strong Convexity

In this section, we analyze the Hessian of objective (1).

Theorem 1 (Local Positive Definiteness). *Let $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ be sampled from the MLR model (3). Let $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ be independent of the samples and lie in the neighborhood of the optimal solution, i.e.,*

$$\|\Delta \boldsymbol{w}_k\| := \|\boldsymbol{w}_k - \boldsymbol{w}_k^*\| \leq c_m \Delta_{\min}, \forall k \in [K], \tag{4}$$

where $c_m = O(p_{\min}(3K)^{-K}(\Delta_{\min}/\Delta_{\max})^{2K-2})$, $\Delta_{\min} = \min_{j \neq k}\{\|\boldsymbol{w}_j^* - \boldsymbol{w}_k^*\|\}$ and $\Delta_{\max} = \max_{j \neq k}\{\|\boldsymbol{w}_j^* - \boldsymbol{w}_k^*\|\}$. Let $P \geq 1$ be a constant. Then if $N \geq O((PK)^K d \log^{K+2}(d))$, w.p. $1 - O(Kd^{-P})$, we have,

$$\frac{1}{8} p_{\min} N \Delta_{\min}^{2K-2} I \preceq \nabla^2 f(\boldsymbol{w} + \delta\boldsymbol{w}) \preceq 10 N (3K)^K \Delta_{\max}^{2K-2} I, \tag{5}$$

for any $\delta\boldsymbol{w} := [\delta\boldsymbol{w}_1; \delta\boldsymbol{w}_2; \cdots; \delta\boldsymbol{w}_K]$ satisfying $\|\delta\boldsymbol{w}_k\| \leq c_f \Delta_{\min}$, where $c_f = O(p_{\min}(3K)^{-K} d^{-K+1}(\Delta_{\min}/\Delta_{\max})^{2K-2})$.

The above theorem shows the Hessians of a small neighborhood around a fixed $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$, which is close enough to the optimum, are positive definiteness (PD). The conditions on $\{\boldsymbol{w}_k\}_{k=1,\cdots,K}$ and $\{\delta\boldsymbol{w}_k\}_{k=1,\cdots,K}$ are different. $\{\boldsymbol{w}_k\}_{k=1,\cdots,K}$ are required to be independent of samples and in a ball of radius $c_m \Delta_{\min}$ centered at the optimal solution. On the other hand, $\{\delta\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ can be dependent on the samples but are required to be in a smaller ball of radius $c_f \Delta_{\min}$. The conditions are natural as if $\Delta_{\min}$ is very small then distinguishing between $\boldsymbol{w}_k^*$ and $\boldsymbol{w}_{k'}^*$ is not possible and hence Hessians will not be PD w.r.t both the components.

To prove the theorem, we decompose the Hessian of Eq. (1) into multiple blocks, $(\nabla f)_{jl} = \frac{\partial^2 f}{\partial \boldsymbol{w}_j \partial \boldsymbol{w}_l} \in \mathbb{R}^{d \times d}$. When $\boldsymbol{w}_k \to \boldsymbol{w}_k^*$ for all $k \in [K]$, the diagonal blocks of the Hessian will be strictly positive definite. At the same time, the off-diagonal blocks will be close to zeros. The blocks are approximated by the samples using matrix Bernstein inequality. The detailed proof can be found in Appendix A.2.

Traditional analysis of optimization methods on strongly convex functions, such as gradient descent, requires the Hessians of all the parameters are PD. Theorem 1 implies that when $\boldsymbol{w}_k = \boldsymbol{w}_k^*$ for all $k = 1, 2, \cdots, K$, a small basin of attraction around the optimum is strongly convex as formally stated in the following corollary.

Corollary 1 (Strong Convexity near the Optimum). *Let $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ be sampled from the MLR model (3). Let $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ lie in the neighborhood of the optimal solution, i.e.,*

$$\|\boldsymbol{w}_k - \boldsymbol{w}_k^*\| \leq c_f \Delta_{\min}, \forall k \in [K], \tag{6}$$

where $c_f = O(p_{\min}(3K)^{-K} d^{-K+1}(\Delta_{\min}/\Delta_{\max})^{2K-2})$. Then, for any constant $P \geq 1$, if $N \geq O((PK)^K d \log^{K+2}(d))$, w.p. $1 - O(Kd^{-P})$, the objective function $f(\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_K)$ in Eq. (1) is strongly convex. In particular, w.p. $1 - O(Kd^{-P})$, for all $\boldsymbol{w}$ satisfying Eq. (6),

$$\frac{1}{8} p_{\min} N \Delta_{\min}^{2K-2} I \preceq \nabla^2 f(\boldsymbol{w}) \preceq 10 N (3K)^K \Delta_{\max}^{2K-2} I. \tag{7}$$

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The strong convexity of Corollary 1 only holds in the basin of attraction near the optimum that has diameter in the order of $O(d^{-K+1})$, which is too small to be achieved by our initialization method (in Sec. 3.2) using $\tilde{O}(d)$ samples. Next, we show by a simple construction, the linear convergence of gradient descent (GD) with resampling is still guaranteed when the solution is initialized in a much larger neighborhood.

Theorem 2 (Convergence of Gradient Descent). *Let $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ be sampled from the MLR model* (3). *Let $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ be independent of the samples and lie in the neighborhood of the optimal solution, defined in Eq.* (4). *One iteration of gradient descent can be described as, $\boldsymbol{w}^+ = \boldsymbol{w} - \eta \nabla f(\boldsymbol{w})$, where $\eta = 1/(10N(3K)^K \Delta_{\max}^{2K-2})$. Then, if $N \geq O(K^K d \log^{K+2}(d))$, w.p. $1 - O(Kd^{-2})$,*

$$\|\boldsymbol{w}^+ - \boldsymbol{w}^*\|^2 \leq (1 - \frac{p_{\min}\Delta_{\min}^{2K-2}}{80(3K)^K \Delta_{\max}^{2K-2}})\|\boldsymbol{w} - \boldsymbol{w}^*\|^2 \tag{8}$$

Remark. The linear convergence Eq. (8) requires the resampling of the data points for each iteration. In Sec. 3.3, we combine Corollary 1, which doesn't require resampling when the iterate is sufficiently close to the optimum, to show that there exists an algorithm using a finite number of samples to achieve any solution precision.

To prove Theorem 2, we prove the PD properties on a line between a current iterate and the optimum by constructing a set of anchor points and then apply traditional analysis for the linear convergence of gradient descent. The detailed proof can be found in Appendix A.3.

3.2 Initialization via Tensor method

In this section, we propose a tensor method to initialize the parameters. We define the second-order moment $M_2 := \mathbb{E}\big[\![y^2(\boldsymbol{x} \otimes \boldsymbol{x} - I)]\!\big]$ and the third-order moments, $M_3 := \mathbb{E}\big[\![y^3 \boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}]\!\big] - \sum_{j\in[d]} \mathbb{E}\big[\![y^3(\boldsymbol{e}_j \otimes \boldsymbol{x} \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_j \otimes \boldsymbol{x} + \boldsymbol{x} \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j)]\!\big]$. According to Lemma 6 in [24], $M_2 = \sum_{k=[K]} 2p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*$ and $M_3 = \sum_{k=[K]} 6p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*$. Therefore by calculating the eigen-decomposition of the estimated moments, we are able to recover the parameters to any precision provided enough samples. Theorem 8 of [24] needs $O(d^3)$ sample complexity to obtain the model parameters with certain precision. Such high sample complexity comes from the tensor concentration bound. However, we find the problem of tensor eigendecomposition in MLR can be reduced to $\mathbb{R}^{K\times K\times K}$ space such that the sample complexity and computational complexity are $O(\text{poly}(K))$. Our method is similar to the whitening process in [7, 19]. However, [7] needs $O(d^6)$ sample complexity due to the nuclear-norm minimization problem, while ours requires only $\widetilde{O}(d)$. For this sample complexity, we need assume the following,

Assumption 1. *The following quantities, $\sigma_K(M_2)$, $\|M_2\|$, $\|M_3\|_{op}^{2/3}$, $\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^2$ and $(\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^3)^{2/3}$, have the same order of d, i.e., the ratios between any two of them are independent of d.*

The above assumption holds when $\{\boldsymbol{w}_k^*\}_{k=1,2,\cdots,K}$ are orthonormal to each other.

We formally present the tensor method in Algorithm 1 and its theoretical guarantee in Theorem 3.

Theorem 3. *Under Assumption 1, if $|\Omega| \geq O(d\log^2(d) + \log^4(d))$, then w.p. $1 - O(d^{-2})$, Algorithm 1 will output $\{\boldsymbol{w}_k^{(0)}\}_{k=1}^K$ that satisfies,*

$$\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| \leq c_m \Delta_{\min}, \forall k \in [K]$$

which falls in the locally PD region, Eq. (4)*, in Theorem 1.*

The proof can be found in Appendix B.2. Forming $\hat{M}_2$ explicitly will cost $O(Nd^2)$ time, which is expensive when d is large. We can compute each step of the power method without explicitly forming $\hat{M}_2$. In particular, we alternately compute $\hat{Y}^{(t+1)} = \sum_{i\in\Omega_{M_2}} y_i^2(\boldsymbol{x}_i(\boldsymbol{x}_i^T Y^{(t)}) - Y^{(t)})$ and let $Y^{(t+1)} = \text{QR}(\hat{Y}^{(t+1)})$. Now each power method iteration only needs $O(KNd)$ time. Furthermore, the number of iterations needed will be a constant, since power method has linear convergence rate and we don't need very accurate solution. For the proof of this claim, we refer

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Algorithm 1 Initialization for MLR via Tensor Method

Input: $\{\boldsymbol{x}_i, y_i\}_{i \in \Omega}$

Output: $\{\boldsymbol{w}_k^{(0)}\}_{k=1}^K$

1: Partition the dataset Ω into $\Omega = \Omega_{M_2} \cup \Omega_2 \cup \Omega_3$ with $|\Omega_{M_2}| = O(d \log^2(d)), |\Omega_2| = O(d \log^2(d))$ and $|\Omega_3| = O(\log^4(d))$

2: Compute the approximate top-K eigenvectors, $Y \in \mathbb{R}^{d \times K}$, of the second-order moment, $\hat{M}_2 := \frac{1}{|\Omega_{M_2}|} \sum_{i \in \Omega_{M_2}} y_i^2 (\boldsymbol{x}_i \otimes \boldsymbol{x}_i - I)$, by the power method.

3: Compute $\hat{R}_2 = \frac{1}{2|\Omega_2|} \sum_{i \in \Omega_2} y_i^2 (Y^T \boldsymbol{x}_i \otimes Y^T \boldsymbol{x}_i - I)$.

4: Compute the whitening matrix $\hat{W} \in \mathbb{R}^{K \times K}$ of $\hat{R}_2$, i.e., $\hat{W} = \hat{U}_2 \hat{\Lambda}_2^{-1/2} \hat{U}_2^T$, where $\hat{R}_2 = \hat{U}_2 \hat{\Lambda}_2 \hat{U}_2^T$ is the eigendecomposition of $\hat{R}_2$.

5: Compute $\hat{R}_3 = \frac{1}{6|\Omega_3|} \sum_{i \in \Omega_3} y_i^3 (\boldsymbol{r}_i \otimes \boldsymbol{r}_i \otimes \boldsymbol{r}_i - \sum_{j \in [K]} \boldsymbol{e}_j \otimes \boldsymbol{r}_i \otimes \boldsymbol{e}_j - \sum_{j \in [K]} \boldsymbol{e}_j \otimes \boldsymbol{e}_j \otimes \boldsymbol{r}_i - \sum_{j \in [K]} \boldsymbol{r}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j)$, where $\boldsymbol{r}_i = Y^T \boldsymbol{x}_i$ for all $i \in \Omega_3$.

6: Compute the eigenvalues $\{\hat{a}_k\}_{k=1}^K$ and the eigenvectors $\{\hat{\boldsymbol{v}}_k\}_{k=1}^K$ of the whitened tensor $\hat{R}_3(\hat{W}, \hat{W}, \hat{W}) \in \mathbb{R}^{K \times K \times K}$ by using the robust tensor power method [2].

7: Return the estimation of the models, $\boldsymbol{w}_k^{(0)} = Y(\hat{W}^T)^\dagger (\hat{a}_k \hat{\boldsymbol{v}}_k)$

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to the proof of Lemma 10 in Appendix B. Next we compute $\hat{R}_2$ using $O(KNd)$ and compute $\hat{W}$ in $O(K^3)$ time. Computing $\hat{R}_3$ takes $O(KNd + K^3 N)$ time. The robust tensor power method takes $O(\text{poly}(K)\text{polylog}(d))$ time. In summary, the computational complexity for the initialization is $O(KdN + K^3 N + \text{poly}(K)\text{polylog}(d)) = \widetilde{O}(dN)$.

3.3 Global Convergence Algorithm

We are now ready to show the complete algorithm, Algorithm 2, that has global convergence guarantee. We use $f_\Omega(\boldsymbol{w})$ to denote the objective function Eq. (1) generated from a subset of the dataset Ω, i.e., $f_\Omega(\boldsymbol{w}) = \sum_{i \in \Omega} \Pi_{k=1}^K (y_i - \boldsymbol{x}_i^T \boldsymbol{w}_k)^2$.

Theorem 4 (Global Convergence Guarantee). *Let $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ be sampled from the MLR model* (3) *with $N \geq O(d(K \log(d))^{2K+3})$. Let the step size η be smaller than a positive constant. Then given any precision $\epsilon > 0$, after $T = O(\log(d/\epsilon))$ iterations, w.p. $1 - O(Kd^{-2} \log(d))$, the output of Algorithm 2 satisfies*

$$\|\boldsymbol{w}^{(T)} - \boldsymbol{w}^*\| \leq \epsilon \Delta_{\min}.$$

The detailed proof is in Appendix B.3. The computational complexity required by our algorithm is near-optimal: (a) tensor method (Algorithm 1) is carefully employed such that only $O(dN)$ computation is needed; (b) gradient descent with resampling is conducted in $\log(d)$ iterations to push the iterate to the next phase; (c) gradient descent without resampling is finally executed to achieve any precision with $\log(1/\epsilon)$ iterations. Therefore the total computational complexity is $O(dN \log(d/\epsilon))$. As shown in the theorem, our algorithm can achieve any precision $\epsilon > 0$ without any sample complexity dependency on ϵ. This follows from Corollary 1 that shows local strong convexity of objective (1) with a fixed set of samples. By contrast, tensor method [7, 24] requires $O(1/\epsilon^2)$ samples and EM algorithm requires $O(\log(1/\epsilon))$ samples[4, 29].

4 Subspace Clustering (SC)

The mixed linear regression problem can be viewed as clustering N $(d+1)$-dimensional data points, $\boldsymbol{z}_i = [\boldsymbol{x}_i, y_i]^T$, into one of the K subspaces, $\{\boldsymbol{z} : [\boldsymbol{w}_k^*, -1]^T \boldsymbol{z} = 0\}$ for $k \in [K]$. Assume we have data points $\{\boldsymbol{z}_i\}_{i=1,2,\cdots,N}$ sampled from the following model,

$$a_i \sim \text{multinomial}(\boldsymbol{p}), \ \boldsymbol{s}_i \sim \mathcal{N}(0, I_r), \ \boldsymbol{z}_i = U_{a_i}^* \boldsymbol{s}_i, \tag{9}$$

where $\boldsymbol{p}$ is the proportion of samples from different subspaces and satisfies $\boldsymbol{p}^T \boldsymbol{1} = 1$ and $\{U_k^*\}_{k=1,2,\cdots,K}$ are the bases of the ground truth subspaces. We can solve Eq. (2) by alternately minimizing over U_k when fixing the others, which is equivalent to finding the top-r eigenvectors

of $\sum_{i=1}^{N} \alpha_i^k \boldsymbol{z}_i \boldsymbol{z}_i^T$, where $\alpha_i^k = \Pi_{j \neq k} \langle I_d - U_j U_j^T, \boldsymbol{z}_i \boldsymbol{z}_i^T \rangle$. When the dimension is high, it is very expensive to compute the exact top-r eigenvectors. A more efficient way is to use one iteration of the power method (aka subspace iteration), which only takes $O(KdN)$ computational time per iteration. We present our algorithm in Algorithm 3.

We show Algorithm 3 will converge to the ground truth when the initial subspaces are sufficiently close to the underlying subspaces. Define $D(\hat{U}, \hat{V}) := \frac{1}{\sqrt{2}} \|UU^T - VV^T\|_F$ for some $\hat{U}, \hat{V} \in \mathbb{R}^{d \times r}$, where U, V are orthogonal bases of $Span(\hat{U}), Span(\hat{V})$ respectively. Define $D_{\max} := \max_{j \neq q} D(U_q^*, U_j^*)$, $D_{\min} := \min_{j \neq q} D(U_q^*, U_j^*)$.

Theorem 5. *Let $\{\boldsymbol{z}_i\}_{i=1,2,\cdots,N}$ be sampled from subspace clustering model (9). If $N \geq O(r(K \log(r))^{2K+2})$ and the initial parameters $\{U_k^0\}_{k \in [K]}$ satisfy*

$$\max_k \{D(U_k^*, U_k^0)\} \leq c_s D_{\min}, \tag{10}$$

where $c_s = O(p_{\min}/p_{\max}(3K)^{-K}(D_{\min}/D_{\max})^{2K-3})$, then w.p. $1 - O(Kr^{-2})$, the sequence $\{U_1^t, U_2^t, \cdots, U_K^t\}_{t=1,2,\cdots}$ generated by Algorithm 3 converges to the ground truth superlinearly. In particular, for $\Delta_t := \max_k \{D(U_k^*, U_k^t)\}$,

$$\Delta_{t+1} \leq \Delta_t^2/(2c_s D_{\min}) \leq \frac{1}{2} \Delta_t.$$

We refer to Appendix C.2 for the proof. Compared to other methods, our sample complexity only depends on the dimension of each subspace linearly. We refer to Table 1 in [23] for a comparison of conditions for different methods. Note that if $D_{\min}/D_{\max}$ is independent of r or d, then the initialization radius c_s is a constant. However, initialization within the required distance to the optima is still an open question; tensor methods do not apply in this case. Interestingly, our experiments seem to suggest that our proposed method converges to the global optima (in the setting considered in the above theorem).

Algorithm 2 Gradient Descent for MLR	**Algorithm 3** Power Method for SC
Input: $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$, step size η.	**Input:** data points $\{\boldsymbol{z}_i\}_{i=1,2,\cdots,N}$
Output: w	**Output:** $\{U_k\}_{k \in [K]}$
1: Partition the dataset into $\{\Omega^{(t)}\}_{t=0,1,\cdots,T_0+1}$	1: Some initialization, $\{U_k^0\}_{k \in [K]}$.
2: Initialize $\boldsymbol{w}^{(0)}$ by Algorithm 1 with $\Omega^{(0)}$	2: Partition the data into $\{\Omega^{(t)}\}_{t=0,1,2,\cdots,T}$.
3: **for** $t = 1, 2, \cdots, T_0$ **do**	3: **for** $t = 0, 1, 2, \cdots, T$ **do**
4: $\quad \boldsymbol{w}^{(t)} = \boldsymbol{w}^{(t-1)} - \eta \nabla f_{\Omega^{(t)}}(\boldsymbol{w}^{(t-1)})$	4: $\quad \alpha_i = \Pi_{j=1}^{K} \langle I_d - U_j^t U_j^{tT}, \boldsymbol{z}_i \boldsymbol{z}_i^T \rangle, i \in \Omega^{(t)}$
5: **for** $t = T_0+1, T_0+2, \cdots, T_0+T_1$ **do**	5: $\quad$ **for** $k = 1, 2, \cdots, K$ **do**
6: $\quad \boldsymbol{w}^{(t)} = \boldsymbol{w}^{(t-1)} - \eta \nabla f_{\Omega^{(T_0+1)}}(\boldsymbol{w}^{(t-1)})$	6: $\quad\quad \alpha_i^k = \alpha_i / \langle I_d - U_k^t U_k^{tT}, \boldsymbol{z}_i \boldsymbol{z}_i^T \rangle$
	7: $\quad\quad U_k^{t+1} \leftarrow QR(\sum_{i \in \Omega^{(t)}} \alpha_i^k \boldsymbol{z}_i \boldsymbol{z}_i^T U_k^t)$

---

5 Numerical Experiments

5.1 Mixed Linear Regression

In this section, we use synthetic data to show the properties of our algorithm that minimizes Eq. (1), which we call *LOSCO* (LOcally Strongly Convex Objective). We generate data points and parameters from standard normal distribution. We set $K = 3$ and $p_k = \frac{1}{3}$ for all $k \in [K]$. The error is defined as $\epsilon^{(t)} = \min_{\pi \in \text{Perm}([K])} \{\max_{k \in [K]} \|\boldsymbol{w}_{\pi(k)}^{(t)} - \boldsymbol{w}_k^*\|/\|\boldsymbol{w}_k^*\|\}$, where $\text{Perm}([K])$ is the set of all the permutation functions on the set $[K]$. The errors reported in the paper are averaged over 10 trials. In our experiments, we find there is no difference whether doing resampling or not. Hence, for simplicity, we use the original dataset for all the processes. We set both of two parameters in the robust tensor power method (denoted as N and L in Algorithm 1 in [2]) to be 100. The experiments are conducted in Matlab. After the initialization, we use alternating minimization (i.e., block coordinate descent) to exactly minimize the objective over $\boldsymbol{w}_k$ for $k = 1, 2, \cdots, K$ cyclicly.

Fig. 1(a) shows the recovery rate for different dimensions and different samples. We call the result of a trial is a successful recovery if $\epsilon^{(t)} < 10^{-6}$ for some $t < 100$. The recovery rate is the proportion of

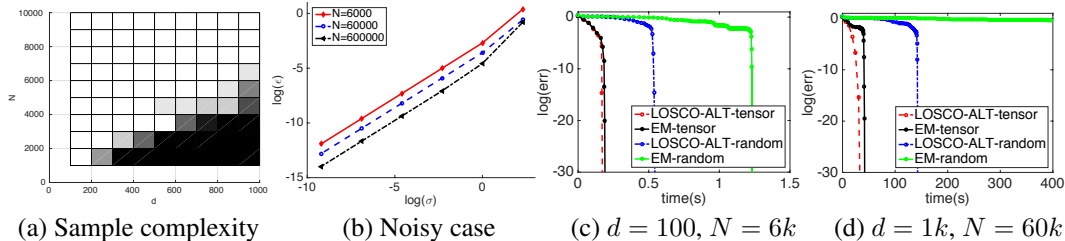|			
(a) Sample complexity	(b) Noisy case	(c) $d = 100$, $N = 6k$	(d) $d = 1k$, $N = 60k$

Figure 1: (a),(b): Empirical performance of our method. (c), (d): performance of our methods vs EM method. Our method with random initialization is signficantly better than EM with random initialization. Performance of the two methods is comparable when initialized with tensor method.

10 trials with successful recovery. As shown in the figure, the sample complexity for exact recovery is nearly linear to d. Fig. 1(b) shows the behavior of our algorithm in the noisy case. The noise is drawn from $e_i \in \mathcal{N}(0, \sigma^2)$, i.i.d., and d is fixed as 100. As we can see from the figure, the solution error is almost proportional to the noise deviation. Comparing among different N's, the solution error decreases when N increases, so it seems consistent in presence of unbiased noise. We also illustrate the performance of our tensor initialization method in Fig. 2(a) in Appendix D.

We next compare with EM algorithm [29], where we alternately assign labels to points and exactly solve each model parameter according to the labels. EM has been shown to be very sensitive to the initialization [29]. The grid search initialization method proposed in [29] is not feasible here, because it only handles two components with a same magnitude. Therefore, we use random initialization and tensor initialization for EM. We compare our method with EM on convergence speed under different dimensions and different initialization methods. We use exact alternating minimization (LOSCO-ALT) to optimize our objective (1), which has similar computational complexity as EM. Fig. 1(c)(d) shows our method is competitive with EM on computational time, when it converges to the optima. In the case of (d), EM with random initialization doesn't converge to the optima, while our method still converges. In Appendix D, we will show some more experimental results.

Table 1: Time (sec.) comparison for different subspace clustering methods

N/K	SSC	SSC-OMP	LRR	TSC	NSN+spectral	NSN+GSR	**PSC**
200	22.08	31.83	4.01	2.76	3.28	5.90	0.41
400	152.61	60.74	11.18	8.45	11.51	15.90	0.32
600	442.29	99.63	33.36	30.09	36.04	33.26	0.60
800	918.94	159.91	79.06	75.69	85.92	54.46	0.73
1000	1738.82	258.39	154.89	151.64	166.70	83.96	0.76

5.2 Subspace Clustering

In this section, we compare our subspace clustering method, which we call PSC (Power method for Subspace Clustering), with state-of-the-art methods, SSC [13], SSC-OMP [12], LRR [22], TSC [17], NSN+spectral [23] and NSN+GSR [23] on computational time. We fix $K = 5$, $r = 30$ and $d = 50$. The ground truth U_k^* is generated from Gaussian matrices. Each data point is a normalized Gaussian vector in their own subspace. Set $p_k = 1/K$. The initial subspace estimation is generated by orthonormlizing Gaussian matrices. The stopping criterion for our algorithm is that every point is clustered correctly, i.e., the clustering error (CE) (defined in [23]) is zero. We use publicly available codes for all the other methods (see [23] for the links).

As we shown from Table 1, our method is much faster than all other methods especially when N is large. Almost all CE's corresponding to the results in Table 1 are very small, which are listed in Appendix D. We also illustrate CE's of our method for different N, d and r when fixing $K = 5$ in Fig. 6 of Appendix D, from which we see whatever the ambient dimension d is, the clusters will be exactly recovered when N is proportional to r.

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Appendix

A Proofs of Local Convergence

We define a sequence of constants, $\{C_J\}_{J=0,1,\cdots}$, that satisfy

$$C_0 = 1, C_1 = 3, \text{ and } C_J = C_{J-1} + (4J^2 + 2J)C_{J-2} \text{ for } J \geq 2. \tag{11}$$

By construction, we can upper bound C_J,

$$
\begin{aligned}
C_J &\leq C_{J-2} + (4J^2 + 2J)C_{J-2} + (4(J-1)^2 + 2J - 2)C_{J-3} \\
&\leq C_{J-2} + (4J^2 + 2J)C_{J-2} + (4(J-1)^2 + 2J - 2)C_{J-2} \\
&\leq 8J^2 C_{J-2} \\
&\leq (3J)^J.
\end{aligned}
\tag{12}
$$

A.1 Some Lemmata

We first introduce some lemmata, whose proofs can be found in Sec. A.4.

Lemma 1 (Proposition 1.1 in [20]). *If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $\Sigma \in \mathbb{R}^{d \times d}$ is a fixed positive semi-definite matrix, then for all $t > 0$, w. p. $1 - e^{-t}$, we have*

$$\boldsymbol{x}^T \Sigma \boldsymbol{x} \leq \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)t} + 2\|\Sigma\|t.$$

By taking $t = P \log(d) + \log(n)$ for some $n \geq d$ and some constant $P \geq 1$, we have the following corollary.

Corollary 2. *If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $\Sigma \in \mathbb{R}^{d \times d}$ is a fixed positive semi-definite matrix, then for a fixed positive constant $P \geq 1$, we have, w. p. $1 - \frac{1}{n}d^{-P}$,*

$$\boldsymbol{x}^T \Sigma \boldsymbol{x} \leq \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)(P\log(d) + \log(n))} + 2\|\Sigma\|(P\log(d) + \log(n)) \leq (4P+5)\operatorname{tr}(\Sigma)\log(n).$$

Setting $\Sigma = \boldsymbol{\beta}\boldsymbol{\beta}^T$ in Corollary 2, we have the following corollary.

Corollary 3. *If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $P \geq 1$ is a constant, then given any fixed $\boldsymbol{\beta} \in \mathbb{R}^d$, w. p. $1 - \frac{1}{n}d^{-P}$, we have*

$$(\boldsymbol{\beta}^T \boldsymbol{x})^2 \leq (4P+5)\|\boldsymbol{\beta}\|^2 \log n.$$

Setting $\Sigma = I$ in Corollary 2, we have the following corollary.

Corollary 4. *If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $P \geq 1$ is a constant, then w. p. $1 - \frac{1}{n}d^{-P}$, we have*

$$\|\boldsymbol{x}\|^2 \leq (4P+5)d\log n.$$

Lemma 2 (Stein-type Lemma). *Let $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $f(\boldsymbol{x})$ be a function of $\boldsymbol{x}$ whose second derivative exists. Then*
$$\mathbb{E}\llbracket f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T \rrbracket = \mathbb{E}\llbracket f(\boldsymbol{x})\rrbracket I + \mathbb{E}\llbracket \nabla^2 f(\boldsymbol{x})\rrbracket$$

Lemma 3. *Let $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$, then*

$$\Pi_{k=1}^K \operatorname{tr}(A_k)I \preceq \mathbb{E}\llbracket \Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T \rrbracket \preceq C_K \Pi_{k=1}^K \operatorname{tr}(A_k)I, \tag{13}$$

where C_K is a constant depending only on K, which is defined in Eq. (11).

Lemma 4. *Let $\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{0}, I_d)$ i.i.d., for all $i \in [n]$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$. Let $B := \mathbb{E}\llbracket \Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T \rrbracket$, $B_i := \Pi_{k=1}^K (\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\boldsymbol{x}_i \boldsymbol{x}_i^T$ and $\hat{B} = \frac{1}{n}\sum_{i=1}^n B_i$.*

If $n \geq O(\frac{1}{\delta^2}\log^K(\frac{1}{\delta})(PK)^K d \log^{K+1} d)$ and $\delta > \frac{\sqrt{4KC_{2K+1}}}{\sqrt{nd^P}}$ for some $0 < \delta \leq 1$ and $P \geq 1$, then w.p. $1 - O(Kd^{-P})$, we have

$$\|\hat{B} - B\| \leq \delta\|B\|. \tag{14}$$

Lemma 5. *Let $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, I_d)$. Then given $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$, we have*

$$\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) \leq \|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\| \leq \sqrt{3C_{2K+1}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k)I. \tag{15}$$

Lemma 6. *Let $\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{0}, I_d)$ i.i.d., for all $i \in [n]$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$. Let $B := \mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]$, $B_i := (\boldsymbol{\beta}^T\boldsymbol{x}_i)(\boldsymbol{\gamma}^T\boldsymbol{x}_i)\Pi_{k=1}^K(\boldsymbol{x}_i^TA_k\boldsymbol{x}_i)\boldsymbol{x}_i\boldsymbol{x}_i^T$ and $\hat{B} = \frac{1}{n}\sum_{i=1}^n B_i$.*

If $n \geq O(\frac{1}{\delta^2}\log^{K+1}(1/\delta)(PK)^Kd\log^{K+2}(d))$, $\delta > \frac{\sqrt{8KC_{2K+3}}}{\sqrt{n}d^P}$ for some $0 < \delta \leq 1$ and $P \geq 1$, then w.p. $1 - O(Kd^{-P})$, we have

$$\|\hat{B} - B\| \leq \delta\|B\|. \tag{16}$$

Lemma 7. *If $n \geq c\log^{K+1}(c)K^{4K}d\log^{K+2}(d)$, where c is a constant, then $n \geq cd\log d\log^{K+1}(n)$.*

A.2 Proof of Theorem 1

Proof. Denote the Hessian of Eq. (1), $H \in \mathbb{R}^{Kd \times Kd}$. Let $H = \sum_i H_i$, where

$$H_i := \begin{bmatrix} H_i^{11} & H_i^{12} & \cdots & H_i^{1K} \\ H_i^{21} & H_i^{22} & \cdots & H_i^{2K} \\ & & \ddots & \\ H_i^{K1} & H_i^{K2} & \cdots & H_i^{KK} \end{bmatrix} \tag{17}$$

For diagonal blocks,

$$H_i^{jj} := 2\big(\Pi_{k\neq j}(y_i - (\boldsymbol{w}_k + \delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \tag{18}$$

For off-diagonal blocks,

$$H_i^{jl} := 4(y_i - (\boldsymbol{w}_j + \delta\boldsymbol{w}_j)^T\boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l + \delta\boldsymbol{w}_l)^T\boldsymbol{x}_i)\big(\Pi_{k\neq j, k\neq l}(y_i - (\boldsymbol{w}_k + \delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \tag{19}$$

In the following we will show that when $\boldsymbol{w}_k$ is close to the optimal solution $\boldsymbol{w}_k^*$ and $\delta\boldsymbol{w}_k$ is small enough for all k, then H will be positive definite w.h.p..

The main idea is to upper bound the off-diagonal blocks and lower bound the diagonal blocks because,

$$\begin{aligned}
\sigma_{min}(H) &= \min_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^K \boldsymbol{a}_j^T H^{jj}\boldsymbol{a}_j + \sum_{j\neq l} 2\boldsymbol{a}_j^T H^{jl}\boldsymbol{a}_l \\
&\geq \min_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^K \sigma_{min}(H^{jj})\|\boldsymbol{a}_j\|^2 - \sum_{j\neq l} \|H^{jl}\|\|\boldsymbol{a}_j\|\|\boldsymbol{a}_l\| \\
&\geq \min_j\{\sigma_{min}(H^{jj})\} - \max_{j\neq l}\{\|H^{jl}\|\}(K-1)(\sum_j \|\boldsymbol{a}_j\|) \\
&\geq \min_j\{\sigma_{min}(H^{jj})\} - (K-1)\max_{j\neq l}\{\|H^{jl}\|\}.
\end{aligned} \tag{20}$$

First consider the diagonal blocks. The idea is to decompose the diagonal blocks into two parts. The first one only contains $\boldsymbol{w}$ and doesn't contain $\delta\boldsymbol{w}$, so for this fixed $\boldsymbol{w}$ we apply Lemma 4 to bound this term. The second one depends on $\delta\boldsymbol{w}$. We find an upper bound for this term which only depends on the magnitude of $\delta\boldsymbol{w}$. Therefore, the bound will hold for any qualified $\delta\boldsymbol{w}$. Let's first define

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$\{k_1, k_2, \cdots, k_{K-1}\} = [K] \backslash \{j\}$.

$$H^{jj} \succeq \sum_{i \in S_j} H_i^{jj}$$

$$= \sum_{i \in S_j} 2\big(\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\succeq \sum_{i \in S_j} 2\big((y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 - 2|y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i| \|\delta \boldsymbol{w}_{k_1}\| \|\boldsymbol{x}_i\|\big) \big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\succeq \underbrace{\sum_{i \in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 \big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{F_1}$$

$$\underbrace{- \sum_{i \in S_j} 4\|\Delta \boldsymbol{w}_{jk_1}^* - \Delta \boldsymbol{w}_{k_1}\| \|\delta \boldsymbol{w}_{k_1}\| \|\boldsymbol{x}_i\|^2 \big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{E_1}$$

$$\tag{21}$$

$$F_1 \succeq \sum_{i \in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 (y_i - \boldsymbol{w}_{k_2}^T \boldsymbol{x}_i)^2 \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$- \sum_{i \in S_j} 4(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 \|\Delta \boldsymbol{w}_{jk_2}^* - \Delta \boldsymbol{w}_{k_2}\| \|\delta \boldsymbol{w}_{k_2}\| \|\boldsymbol{x}_i\|^2 \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\succeq \underbrace{\sum_{i \in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 (y_i - \boldsymbol{w}_{k_2}^T \boldsymbol{x}_i)^2 \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{F_2}$$

$$\underbrace{- \sum_{i \in S_j} 4\|\Delta \boldsymbol{w}_{jk_1}^* - \Delta \boldsymbol{w}_{k_1}\|^2 \|\Delta \boldsymbol{w}_{jk_2}^* - \Delta \boldsymbol{w}_{k_2}\| \|\delta \boldsymbol{w}_{k_2}\| \|\boldsymbol{x}_i\|^4 \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{E_2}$$

$$\tag{22}$$

Similarly, we decompose $F_n = F_{n+1} - E_{n+1}$, for $n = 1, 2, \cdots, K-1$. Then, recursively, we have

$$H^{jj} \succeq F_1 - E_1 \succeq F_2 - E_2 - E_1 \succeq \cdots \succeq F_{K-1} - E_{K-1} - E_{K-2} - \cdots - E_1 \tag{23}$$

So H^{jj} is decomposed into F_{K-1}, which contains only $\boldsymbol{w}$, and $E_1, E_2, \cdots, E_{K-1}$, each of which contains a separate term of $\|\delta \boldsymbol{w}\|$.

By Lemma 3 and Lemma 4,

$$E_1 \preceq 4 \sum_{i \in S_j} \|\Delta \boldsymbol{w}_{jk_1}^* - \Delta \boldsymbol{w}_{k_1}\| \|\delta \boldsymbol{w}_{k_1}\| (\Pi_{s=2}^{K-1} \|\Delta \boldsymbol{w}_{jk_s}^* - \Delta \boldsymbol{w}_{k_s} - \delta \boldsymbol{w}_{k_s}\|^2) \|\boldsymbol{x}_i\|^{2(K-1)} \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq 4 c_f (1 + c_m + c_f)^{2K-3} \Pi_{k:k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2 \sum_{i \in S_j} \|\boldsymbol{x}_i\|^{2(K-1)} \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq 6 c_f (1 + c_m + c_f)^{2K-3} \Pi_{k:k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2 p_j N C_{K-1} d^{K-1} I$$

$$\tag{24}$$

and similarly, for all $r = 1, 2, \cdots, K-1$,

$$E_r \preceq 6 c_f (1 + c_m + c_f)^{2K-3} \Pi_{k:k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2 p_j N C_{K-1} d^{K-1} I. \tag{25}$$

For F_{K-1}, we have

$$F_{K-1} = \sum_{i \in S_j} 2\big(\Pi_{k \neq j}(y_i - \boldsymbol{w}_k^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\overset{\xi_1}{\succeq} p_j N \Pi_{k \neq j} \|\Delta \boldsymbol{w}_{jk}^* - \Delta \boldsymbol{w}_k\|^2 I \tag{26}$$

$$\succeq p_j N \Pi_{k \neq j} (\|\Delta \boldsymbol{w}_{jk}^*\| - \|\Delta \boldsymbol{w}_k\|)^2 I$$

$$\succeq p_j N (1 - c_m)^{2(K-1)} \Pi_{k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2 I$$

where ξ_1 is because of Lemma 3 and Lemma 4 by setting $A_k = (\Delta \boldsymbol{w}^*_{jk} - \Delta \boldsymbol{w}_k)(\Delta \boldsymbol{w}^*_{jk} - \Delta \boldsymbol{w}_k)^T$ and $\delta = 1/(2C_{K-1})$.

Now combining Eq. (26), Eq. (23) and Eq. (25), we can lower bound the eigenvalues of H^{jj},

$$H^{jj} \succeq \left((1 - c_m)^{2(K-1)} - 6c_f(K-1)(1 + c_m + c_f)^{2K-3}C_{K-1}d^{K-1} \right) p_j N \Pi_{k \neq j} \|\Delta \boldsymbol{w}^*_{jk}\|^2 I \tag{27}$$

Next consider the off-diagonal blocks for $j \neq l$,

$$\sum_{i \in S_q} H^{jl}_i$$

$$= \sum_{i \in S_q} 4(y_i - (\boldsymbol{w}_j + \delta \boldsymbol{w}_j)^T \boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l + \delta \boldsymbol{w}_l)^T \boldsymbol{x}_i)\left(\Pi_{k \neq j, k \neq l}(y_i - (\boldsymbol{w}_k + \delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq \sum_{i \in S_q} 4(y_i - \boldsymbol{w}_j^T \boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l + \delta \boldsymbol{w}_l)^T \boldsymbol{x}_i)\left(\Pi_{k \neq j, k \neq l}(y_i - (\boldsymbol{w}_k + \delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$+ \sum_{i \in S_q} 4\|\delta \boldsymbol{w}_j^T \boldsymbol{x}_i\| |y_i - (\boldsymbol{w}_l + \delta \boldsymbol{w}_l)^T \boldsymbol{x}_i| \left(\Pi_{k \neq j, k \neq l}(y_i - (\boldsymbol{w}_k + \delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq \sum_{i \in S_q} 4(y_i - \boldsymbol{w}_j^T \boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T \boldsymbol{x}_i)\left(\Pi_{k \neq j, k \neq l}(y_i - (\boldsymbol{w}_k + \delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$+ \sum_{i \in S_q} 4|y_i - \boldsymbol{w}_j^T \boldsymbol{x}_i| \|\delta \boldsymbol{w}_l^T \boldsymbol{x}_i\| \left(\Pi_{k \neq j, k \neq l}(y_i - (\boldsymbol{w}_k + \delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$+ \sum_{i \in S_q} 4\|\delta \boldsymbol{w}_j\| \|\boldsymbol{w}_q^* - \boldsymbol{w}_l - \delta \boldsymbol{w}_l\| \left(\Pi_{k \neq j, k \neq l}\|\boldsymbol{w}_q^* - \boldsymbol{w}_k + \delta \boldsymbol{w}_k\|^2 \right) \|\boldsymbol{x}_i\|^{2(K-1)} \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq$$

$$\vdots$$

$$\preceq \sum_{i \in S_q} 4(y_i - \boldsymbol{w}_j^T \boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T \boldsymbol{x}_i)\left(\Pi_{k \neq j, k \neq l}(y_i - \boldsymbol{w}_k^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$+ 8(K-1)c_f(1 + c_m + c_f)^{2K-3} \Delta_{max}^{2K-2} \sum_{i \in S_q} \|\boldsymbol{x}_i\|^{2(K-1)} \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq \sum_{i \in S_q} 4(y_i - \boldsymbol{w}_j^T \boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T \boldsymbol{x}_i)\left(\Pi_{k \neq j, k \neq l}(y_i - \boldsymbol{w}_k^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$+ 12(K-1)c_f(1 + c_m + c_f)^{2K-3} \Delta_{max}^{2K-2} p_q N C_{K-1} d^{K-1} I \tag{28}$$

For the first term above,

$$\| \sum_{i \in S_q} 4(\boldsymbol{w}_q^* - \boldsymbol{w}_j)^T \boldsymbol{x}_i (\boldsymbol{w}_q^* - \boldsymbol{w}_l)^T \boldsymbol{x}_i \left(\Pi_{k \neq j, k \neq l}((\boldsymbol{w}_q^* - \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T \|$$

$$\overset{\xi_1}{\leq} 6p_q N \| \mathbb{E}\left[\!\left[(\boldsymbol{w}_q^* - \boldsymbol{w}_j)^T \boldsymbol{x}_i (\boldsymbol{w}_q^* - \boldsymbol{w}_l)^T \boldsymbol{x}_i \left(\Pi_{k \neq j, k \neq l}((\boldsymbol{w}_q^* - \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2 \right) \boldsymbol{x}_i \boldsymbol{x}_i^T \right]\!\right] \| \tag{29}$$

$$\overset{\xi_2}{\leq} 6p_q N \sqrt{3C_{2K-3}} \|\boldsymbol{w}_q^* - \boldsymbol{w}_j\| \|\boldsymbol{w}_q^* - \boldsymbol{w}_l\| \left(\Pi_{k \neq j, k \neq l}\|\boldsymbol{w}_q^* - \boldsymbol{w}_k\|^2 \right)$$

$$\leq 6p_q N \sqrt{3C_{2K-3}} \|\Delta \boldsymbol{w}^*_{qj} - \Delta \boldsymbol{w}_j\| \|\Delta \boldsymbol{w}^*_{ql} - \Delta \boldsymbol{w}_l\| \left(\Pi_{k \neq j, k \neq l}\|\Delta \boldsymbol{w}^*_{qk} - \Delta \boldsymbol{w}_k\|^2 \right),$$

where ξ_1 is because of Lemma 6 and ξ_2 is because of Lemma 5.

We consider three cases: $\{q \neq j, q \neq l\}$, $q = j$ and $q = l$. When $q \neq j$ and $q \neq l$,

$$\|\Delta \boldsymbol{w}^*_{qj} - \Delta \boldsymbol{w}_j\| \|\Delta \boldsymbol{w}^*_{ql} - \Delta \boldsymbol{w}_l\| \left(\Pi_{k \neq j, k \neq l}\|\Delta \boldsymbol{w}^*_{qk} - \Delta \boldsymbol{w}_k\|^2 \right)$$

$$\leq (1 + c_m)^{2K-2} c_m^2 \|\Delta \boldsymbol{w}^*_{qj}\| \|\Delta \boldsymbol{w}^*_{ql}\| \left(\Pi_{k \neq j, k \neq l}\|\Delta \boldsymbol{w}^*_{qk}\|^2 \right) \tag{30}$$

When $q = j$,

$$\|\Delta \boldsymbol{w}_{qj}^* - \Delta \boldsymbol{w}_j\|\|\Delta \boldsymbol{w}_{ql}^* - \Delta \boldsymbol{w}_l\|\left(\Pi_{k \neq j, k \neq l}\|\Delta \boldsymbol{w}_{qk}^* - \Delta \boldsymbol{w}_k\|^2\right)$$
$$\leq (1 + c_m)^{2K-1} c_m \|\Delta \boldsymbol{w}_{qj}^*\|\|\Delta \boldsymbol{w}_{ql}^*\|\left(\Pi_{k \neq j, k \neq l}\|\Delta \boldsymbol{w}_{qk}^*\|^2\right) \tag{31}$$

For $q = l$, we have similar results. Therefore,

$$\begin{aligned}
\|H^{jl}\| &\leq \sum_{q=1}^{K} \|\sum_{i \in S_q} H_i^{jl}\| \\
&\leq \sum_q (1 + c_m)^{2K-1} c_m 6 p_q N \sqrt{3 C_{2K-3}} \Delta_{max}^{2K-2} \\
&\quad + \sum_q 12(K-1) c_f (1 + c_m + c_f)^{2K-3} p_q N C_{K-1} d^{K-1} \Delta_{max}^{2K-2} \\
&\leq (1 + c_m)^{2K-1} c_m 6 N \sqrt{3 C_{2K-3}} \Delta_{max}^{2K-2} \\
&\quad + 12(K-1) c_f (1 + c_m + c_f)^{2K-3} N C_{K-1} d^{K-1} \Delta_{max}^{2K-2}
\end{aligned} \tag{32}$$

Now we obtain the lower bound for the minimal eigenvalue of the Hessian. When $c_m \leq \frac{p_{min} \Delta_{min}^{2K-2}}{500 K \sqrt{C_{2K-3}} \Delta_{max}^{2K-2}}$ and $c_f \leq \frac{p_{min} \Delta_{min}^{2K-2}}{1000(K-1)^2 C_{K-1} d^{K-1} \Delta_{max}^{2K-2}}$, we have $(1 - c_m)^{2K-2} \geq (1 - \frac{1}{2K})^{2K-2} \geq \frac{1}{4}$, $(1 + c_m + c_f)^{2K-2} \leq 3$. Hence,

$$\|H^{jl}\| \leq \frac{1}{16(K-1)} p_{min} N \Delta_{min}^{2K-2}, \tag{33}$$

Combining Eq.(20), Eq.(27) and Eq.(33), we have

$$\sigma_{min}(H) \geq \frac{1}{8} p_{min} N \Delta_{min}^{2K-2}, \tag{34}$$

which is a positive constant.

In the following we upper bound the maximal eigenvalue of the Hessian.

$$\begin{aligned}
\sigma_{max}(H) &= \max_{\sum_{j=1}^{K}\|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^{K} \boldsymbol{a}_j^T H^{jj} \boldsymbol{a}_j + \sum_{j \neq l} 2\boldsymbol{a}_j^T H^{jl} \boldsymbol{a}_l \\
&\leq \max_{\sum_{j=1}^{K}\|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^{K} \|(H^{jj})\|\|\boldsymbol{a}_j\|^2 + \sum_{j \neq l} \|H^{jl}\|\|\boldsymbol{a}_j\|\|\boldsymbol{a}_l\| \\
&\leq \max_j \{\|H^{jj}\|\} + \max_{j \neq l}\{\|H^{jl}\|\}(K-1)(\sum_j \|\boldsymbol{a}_j\|) \\
&\leq \max_j \{\|H^{jj}\|\} + (K-1) \max_{j \neq l}\{\|H^{jl}\|\}.
\end{aligned} \tag{35}$$

Consider the diagonal blocks and define $\{k_1, k_2, \cdots, k_{K-1}\} = [K] \backslash \{j\}$.

$$\begin{aligned}
H_i^{jj} &= 2\left(\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\right) \boldsymbol{x}_i \boldsymbol{x}_i^T \\
&\preceq 2\left((y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 + 2|y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i||\delta \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i| + (\delta \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2\right)\left(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\right) \boldsymbol{x}_i \boldsymbol{x}_i^T \\
&\preceq \underbrace{2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2\left(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\right) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{F}_1} \\
&\quad + \underbrace{2(2\|\Delta \boldsymbol{w}_{jk_1}^* - \Delta \boldsymbol{w}_{k_1}\| + \|\delta \boldsymbol{w}_{k_1}\|)\|\delta \boldsymbol{w}_{k_1}\|\|\boldsymbol{x}_i\|^2\left(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\right) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{E}_1}
\end{aligned} \tag{36}$$

For $\tilde{E}_1$,

$$\tilde{E}_1 \preceq 4 c_f (1 + c_m + c_f)^{2K-3} \Delta_{max}^{2K-2} \|\boldsymbol{x}_i\|^{2K-2} \boldsymbol{x}_i \boldsymbol{x}_i^T \tag{37}$$

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For $\tilde{F}_1$,

$$\tilde{F}_1$$

$$\preceq \underbrace{2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 (y_i - \boldsymbol{w}_{k_2}^T \boldsymbol{x}_i)^2 \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{F}_2}$$

$$+ \underbrace{2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 |(2\Delta \boldsymbol{w}_{jk_2}^* - 2\Delta \boldsymbol{w}_{k_2} - \delta \boldsymbol{w}_{k_2})^T \boldsymbol{x}_i| |\delta \boldsymbol{w}_{k_2}^T \boldsymbol{x}_i| \big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{E}_2}$$

$$(38)$$

We also have for $\tilde{E}_2$

$$\tilde{E}_2 \preceq 4 c_f (1 + c_m + c_f)^{2K-3} \Delta_{max}^{2K-2} \|\boldsymbol{x}_i\|^{2K-2} \boldsymbol{x}_i \boldsymbol{x}_i^T \tag{39}$$

Therefore, recursively, we have

$$H_i^{jj} \preceq \underbrace{2\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2 \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{F}_{K-1}}$$

$$+ 4K c_f (1 + c_m + c_f)^{2K-3} \Delta_{max}^{2K-2} \|\boldsymbol{x}_i\|^{2K-2} \boldsymbol{x}_i \boldsymbol{x}_i^T \tag{40}$$

Now applying Lemma 3 and Lemma 4,

$$H^{jj} = \sum_q \sum_{i \in S_q} H_i^{jj}$$

$$\preceq 6 c_f K (1 + c_m + c_f)^{2K-3} N C_{K-1} d^{K-1} \Delta_{max}^{2K-2} I + \sum_q \sum_{i \in S_q} 2\big(\Pi_{k \neq q}((\Delta \boldsymbol{w}_{jk}^* - \Delta \boldsymbol{w}_k)^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$\preceq 6 c_f K (1 + c_m + c_f)^{2K-3} N C_{K-1} d^{K-1} \Delta_{max}^{2K-2} I + 3 \sum_q p_q N C_{K-1} \big(\Pi_{k \neq q} \|\Delta \boldsymbol{w}_{jk}^* - \Delta \boldsymbol{w}_k\|^2\big)$$

$$= 6 c_f K (1 + c_m + c_f)^{2K-3} N C_{K-1} d^{K-1} \Delta_{max}^{2K-2} I$$

$$\quad + 3 p_j N C_{K-1} (1 + c_m)^{2K-2} \big(\Pi_{k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2\big)$$

$$\quad + 3 \sum_{q: q \neq j} p_q N C_{K-1} c_m^2 (1 + c_m)^{2K-4} \big(\Pi_{k: k \neq j} \|\Delta \boldsymbol{w}_{jk}^*\|^2\big)$$

$$\preceq 9 N C_{K-1} \Delta_{max}^{2K-2} I$$

$$(41)$$

Combining the off-diagonal blocks bound in Eq. (33), applying union bound on the probabilities of the lemmata and Eq. (12) complete the proof. $\qquad \square$

A.3   Proof of Theorem 2

We first introduce a corollary of Theorem 1, which shows the strong convexity on a line between a current iterate and the optimum.

Corollary 5 (Positive Definiteness on the Line between $\boldsymbol{w}$ and $\boldsymbol{w}^*$)**.** *Let* $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ *be sampled from the MLR model* (3)*. Let* $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ *be independent of the samples and lie in the neighborhood of the optimal solution, defined in Eq.* (4)*. Then, if* $N \geq O(K^K d \log^{K+2}(d))$*, w.p.* $1 - O(K d^{-2})$*, for all* $\lambda \in [0, 1]$*,*

$$\frac{1}{8} p_{\min} N \Delta_{\min}^{2K-2} I \preceq \nabla^2 f(\lambda \boldsymbol{w}^* + (1-\lambda)\boldsymbol{w}) \preceq 10 N (3K)^K \Delta_{\max}^{2K-2} I. \tag{42}$$

Proof. We set d^{K-1} anchor points equally along the line $\lambda \boldsymbol{w}^* + (1-\lambda)\boldsymbol{w}$ for $\lambda \in [0, 1]$. Then based on these anchors, according to Theorem 1, by setting $P = K + 1$, we complete the proof. $\qquad \square$

Now we show the proof of Theorem 2.

Proof. Let $\alpha := \frac{1}{8}p_{\min}N\Delta_{\min}^{2K-2}$ and $\beta := 10N(3K)^K\Delta_{\max}^{2K-2}$.

$$
\begin{aligned}
\|\boldsymbol{w}^+ - \boldsymbol{w}^*\|^2 &= \|\boldsymbol{w} - \eta\nabla f(\boldsymbol{w}) - \boldsymbol{w}^*\|^2 \\
&= \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - 2\eta\nabla f(\boldsymbol{w})^T(\boldsymbol{w} - \boldsymbol{w}^*) + \eta^2\|\nabla f(\boldsymbol{w})\|^2
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\nabla f(\boldsymbol{w}) &= \left(\int_0^1 \nabla^2 f(\boldsymbol{w}^* + \gamma(\boldsymbol{w} - \boldsymbol{w}^*))d\gamma\right)(\boldsymbol{w} - \boldsymbol{w}^*) \\
&=: \hat{H}(\boldsymbol{w} - \boldsymbol{w}^*)
\end{aligned}
\tag{44}
$$

According to Corollary 5,

$$
\alpha I \preceq \hat{H} \preceq \beta I.
\tag{45}
$$

$$
\|\nabla f(\boldsymbol{w})\|^2 = (\boldsymbol{w} - \boldsymbol{w}^*)^T\hat{H}^2(\boldsymbol{w} - \boldsymbol{w}^*) \leq \beta(\boldsymbol{w} - \boldsymbol{w}^*)^T\hat{H}(\boldsymbol{w} - \boldsymbol{w}^*)
\tag{46}
$$

Therefore,

$$
\begin{aligned}
\|\boldsymbol{w}^+ - \boldsymbol{w}^*\|^2 &\leq \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - (-\eta^2\beta + 2\eta)(\boldsymbol{w} - \boldsymbol{w}^*)^T\hat{H}(\boldsymbol{w} - \boldsymbol{w}^*) \\
&\leq \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - (-\eta^2\beta + 2\eta)\alpha\|\boldsymbol{w} - \boldsymbol{w}^*\|^2 \\
&= \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - \frac{\alpha}{\beta}\|\boldsymbol{w} - \boldsymbol{w}^*\|^2 \\
&\leq (1 - \frac{\alpha}{\beta})\|\boldsymbol{w} - \boldsymbol{w}^*\|^2
\end{aligned}
\tag{47}
$$

where the third equality holds by setting $\eta = \frac{1}{\beta}$. $\qquad\square$

A.4 Proof of the lemmata

A.4.1 Proof of Lemma 2

Proof. Let $g(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}}e^{-\|\boldsymbol{x}\|^2/2}$ and we have $\boldsymbol{x}g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\mathrm{d}g(\boldsymbol{x})$.

$$
\begin{aligned}
\mathbb{E}[\![f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!] &= \int f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\
&= -\int f(\boldsymbol{x})(\mathrm{d}g(\boldsymbol{x}))\boldsymbol{x}^T \\
&= \int \nabla f(\boldsymbol{x})\boldsymbol{x}^T g(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} + \int f(\boldsymbol{x})g(\boldsymbol{x})I\mathrm{d}\boldsymbol{x} \\
&= -\int \nabla f(\boldsymbol{x})(\mathrm{d}g(\boldsymbol{x}))^T + \mathbb{E}[\![f(\boldsymbol{x})]\!]I \\
&= \mathbb{E}[\![\nabla^2 f(\boldsymbol{x})]\!] + \mathbb{E}[\![f(\boldsymbol{x})]\!]I
\end{aligned}
\tag{48}
$$

$\qquad\square$

A.4.2 Proof of Lemma 3

Proof. Let $G_K := \mathbb{E}[\![\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]$. First we show the lower bound.

$$
\begin{aligned}
\sigma_{min}(G_K) &= \min_{\|\boldsymbol{a}\|=1} \mathbb{E}[\![\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})(\boldsymbol{x}^T\boldsymbol{a})^2]\!] \\
&\geq \Pi_{k=1}^K\mathbb{E}[\![(\boldsymbol{x}^T A_k\boldsymbol{x})]\!]\min_{\|\boldsymbol{a}\|=1}\mathbb{E}[\![(\boldsymbol{x}^T\boldsymbol{a})^2]\!] \\
&= \Pi_{k=1}^K\mathrm{tr}(A_k)
\end{aligned}
\tag{49}
$$

Next, we show the upper bound. As we know, when $K = 1$, $G_1 = \mathrm{tr}(A_1)I + 2A_1$ and for any $K > 1$, G_K should have an explicit closed-form. However, it is too complicated to derive and formulate it for general K. Fortunately we only need the property of Eq. (13) in our proofs. We

prove it by induction. First, it is obvious that Eq. (13) holds for $K = 1$ and $C_1 = 3$. We assume that, for any $J < K$, there exists a constant C_J depending only on J, such that

$$G_J \preceq C_J \Pi_{k=1}^{J} \operatorname{tr}(A_k) I \tag{50}$$

Then by Stein-type lemma, Lemma 2,

$$
\begin{aligned}
G_K =& \mathbb{E}\big[\!\!\big[\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\big]\!\!\big] \\[4pt]
=& \mathbb{E}\big[\!\!\big[\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x})\big]\!\!\big] I + 2\sum_{j=1}^{K} \mathbb{E}\big[\!\!\big[(\Pi_{k\neq j}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x}))A_j\big]\!\!\big] \\[4pt]
& + 4\sum_{j,l:j\neq l} A_j \mathbb{E}\big[\!\!\big[(\Pi_{k:k\neq j,k\neq l}(\boldsymbol{x}^T A_k \boldsymbol{x}))\boldsymbol{x}\boldsymbol{x}^T\big]\!\!\big] A_l \\[4pt]
\preceq& C_{K-1}\Pi_{k=1}^{K} \operatorname{tr}(A_k) I + 2\sum_{j=1}^{K} C_{K-2}(\Pi_{k\neq j}\operatorname{tr}(A_k))A_j \\[4pt]
& + 4\sum_{j,l:j\neq l} C_{K-2}\|A_j\|\|A_l\|\Pi_{k:k\neq j,k\neq l}\operatorname{tr}(A_k) I \\[4pt]
\preceq& \big(C_{K-1} + (2K + 4K^2)C_{K-2}\big)\Pi_{k=1}^{K}\operatorname{tr}(A_k) I
\end{aligned}
\tag{51}
$$

So $C_K = C_{K-1} + (4K^2 + 2K)C_{K-2}$. Note that $C_0 = 1$. $\qquad\square$

A.4.3 Proof of Lemma 4

Proof. Proof Sketch: We use matrix Bernstein inequality to prove this lemma. However, the spectral norm of the random matrix B_i is not uniformly bounded, which is required by matrix Bernstein inequality. So we define a new random matrix,

$$M_i := \mathbf{1}(\mathcal{E}_i)\Pi_{k=1}^{K}(\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\boldsymbol{x}_i \boldsymbol{x}_i^T,$$

where $\mathcal{E}_i$ is an event when $\|B_i\|$ is bounded, which will hold with high probability and $\mathbf{1}()$ is the indicate function of value 1 and 0, i.e., $\mathbf{1}(\mathcal{E}) = 1$ if $\mathcal{E}$ holds and $\mathbf{1}(\mathcal{E}) = 0$ otherwise. Then

$$\|\hat{B} - B\| \leq \|\hat{B} - \hat{M}\| + \|\hat{M} - M\| + \|M - B\|,$$

where $M = \mathbb{E}[\![M_i]\!]$ and $\hat{M} = \frac{1}{n}\sum_{i=1}^{n} M_i$. We show that

1. $\hat{M} = \hat{B}$ w.h.p. by the union bound
2. $\|\hat{M} - M\|$ is bounded by matrix Bernstein inequality
3. $\|M - B\|$ is bounded because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]$ is small.

Proof Details:

Step 1. First we show that $\|B_i\|$ is bounded w.h.p.. First,

$$\|B_i\| = \Pi_{k=1}^{K}(\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\|\boldsymbol{x}_i\|^2$$

Since $\boldsymbol{x} \sim \mathcal{N}(0, I_d)$, by Corollary 4, we have $\mathbb{P}[\![\|\boldsymbol{x}\|^2 \geq (4P + 5)d\log n]\!] \leq n^{-1}d^{-P}$. By Corollary 2, $\mathbb{P}[\![\boldsymbol{x}^T A_k \boldsymbol{x} > (4P + 5)\operatorname{tr}(A_k)\log n]\!] \leq n^{-1}d^{-P}$. Therefore w.p. $1 - (K+1)n^{-1}d^{-P}$,

$$\|B_i\| \leq (4P + 5)^{K+1} \times (\Pi_{k=1}^{K}\operatorname{tr}(A_k))d\log^{K+1}(n).$$

Define

$$m := (4P + 5)^{K+1}(\Pi_{k=1}^{K}\operatorname{tr}(A_k))d\log^{K+1}(n). \tag{52}$$

and the event

$$\mathcal{E}_i = \{\|B_i\| \leq m\},$$

Let $\mathcal{E}^c$ be the complementary set of $\mathcal{E}$, thus $\mathbb{P}[\![\mathcal{E}_i^c]\!] \leq (K+1)n^{-1}d^{-P}$. By union bound, w.p. $1 - (K+1)d^{-P}$, $\|B_i\| \leq m$ for all $i \in [n]$ and $\hat{M} = \hat{B}$.

Step 2. Now we bound $\|\hat{M} - M\|$ by Matrix Bernstein's inequality[26].

Set $Z_i := M_i - M$. Thus $\mathbb{E}[\![Z_i]\!] = 0$ and $\|Z_i\| \leq 2m$. And

$$\|\mathbb{E}[\![Z_i^2]\!]\| = \|\mathbb{E}[\![M_i^2]\!] - M^2\| \leq \|\mathbb{E}[\![M_i^2]\!]\| + \|M^2\|$$

Since M is PSD, $\|\mathbb{E}[\![M_i^2]\!]\| \leq m\|M\|$. Now by matrix Bernstein's inequality, for any $\delta > 0$,

$$\mathbb{P}\left[\!\left[\frac{1}{n}\|\sum_{i=1}^{n} Z_i\| \geq \delta\|M\|\right]\!\right] \leq 2d\exp(-\frac{\delta^2 n^2\|M\|^2/2}{mn\|M\| + 2mn\delta\|M\|/3}) = 2d\exp(-\frac{\delta^2 n\|M\|/2}{m + 2m\delta/3}) \tag{53}$$

Setting

$$n \geq (P+1)(\frac{4}{3\delta} + \frac{2}{\delta^2})m\|M\|^{-1}\log d, \tag{54}$$

we have w.p. at least $1 - 2d^{-P}$,

$$\|\frac{1}{n}\sum M_i - M\| \leq \delta\|M\| \tag{55}$$

Step 3. Now we bound $\|M - B\|$. For simplicity, we replace $\boldsymbol{x}_i$ by $\boldsymbol{x}$ and $\mathcal{E}_i$ by $\mathcal{E}$.

$$
\begin{aligned}
&\|M - B\| \\
=&\|\mathbb{E}[\![B_i\mathbf{1}(\mathcal{E}_i^c)]\!]\| \\
=& \max_{\|\boldsymbol{a}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k\boldsymbol{x})\mathbf{1}(\mathcal{E}^c)]\!] \\
\overset{\zeta_1}{\leq}& \max_{\|\boldsymbol{a}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^4\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k\boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
=& \max_{\|\boldsymbol{a}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, \mathbb{E}[\![(\boldsymbol{x}^T\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{x})\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k\boldsymbol{x})^2\boldsymbol{x}\boldsymbol{x}^T]\!]\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_2}{\leq}& \max_{\|\boldsymbol{a}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, C_{2K+1}\Pi_{k=1}^{K}\operatorname{tr}(A_k)^2 I\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_3}{\leq}& \frac{\sqrt{(K+1)C_{2K+1}}}{\sqrt{nd^P}}\Pi_{k=1}^{K}\operatorname{tr}(A_k)
\end{aligned}
\tag{56}
$$

where ζ_1 is from Holder's inequality, ζ_2 is because of Lemma 3 and ζ_3 is because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!] = \mathbb{P}[\![\mathcal{E}^c]\!]$. Assume $n \geq 4(K+1)C_{2K+1}/d^P$, we have $\|M - B\| \leq \frac{1}{2}\|B\|$ and $\frac{3}{2}\|B\| \geq \|M\| \geq \frac{1}{2}\|B\|$. So combining this result with Eq. (52), Eq. (54), and Eq. (55), if

$$n \geq \max\{4(K+1)C_{2K+1}/d^P, c_1\frac{1}{\delta^2}(4P+5)^{K+2}d\log^{K+1}(n)\log d\}, \tag{57}$$

we obtain

$$\|\frac{1}{n}\sum M_i - M\| \leq \frac{1}{3}\delta\|M\| \leq \frac{1}{2}\delta\|B\|. \tag{58}$$

According to Lemma 7, $n \geq O(\frac{1}{\delta^2}\log^{K+1}(\frac{1}{\delta})(PK)^K d\log^{K+2} d)$ will imply Eq. (57). By further setting $\delta > \frac{\sqrt{4(K+1)C_{2K+1}}}{\sqrt{nd^P}}$, we have $\|M - B\| \leq \frac{1}{2}\delta\|B\|$, completing the proof.

$\square$

A.4.4 Proof of Lemma 5

Proof.

$$
\begin{aligned}
&\|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\| \\
&\geq \mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k\boldsymbol{x})]\!]/(\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|) \\
&\geq \|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^{K}\operatorname{tr}(A_k).
\end{aligned}
\tag{59}
$$

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$$\|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\|$$

$$= \max_{\boldsymbol{a},\boldsymbol{b}} \mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})(\boldsymbol{a}^T\boldsymbol{x})(\boldsymbol{b}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})]\!]/(\|\boldsymbol{a}\|\|\boldsymbol{b}\|)$$

$$\leq \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2(\boldsymbol{b}^T\boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})^2]\!]^{1/2}/(\|\boldsymbol{a}\|\|\boldsymbol{b}\|)$$

$$\leq \sqrt{3C_{2K+1}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) \tag{60}$$

$\square$

A.4.5 Proof of Lemma 6

Proof. Note that the matrix B_i is probably not PSD. Thus we can't apply Lemma 4 directly. But the proof is similar to that for Lemma 4.

Define

$$m := (4P+5)^{K+2}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|(\Pi_{k=1}^K \operatorname{tr}(A_k))d\log^{K+1}(n), \tag{61}$$

and the event, $\mathcal{E}_i := \{\|B_i\| \leq m\}$. Then by Corollary 3,

$$\mathbb{P}[\![\mathcal{E}_i]\!] \geq 1 - 2Kn^{-1}d^{-P}.$$

Define a new random matrix $M_i := \mathbf{1}(\mathcal{E}_i)B_i$, its expectation $M := \mathbb{E}[\![M_i]\!]$ and its empirical average $\hat{M} = \frac{1}{n}\sum_{i=1}^n M_i$.

Step 1. By union bound, we have w.p. $1 - 2Kd^{-P}$, $M_i = B_i$ for all i, i.e., $\hat{M} = \hat{B}$.

Step 2. We now bound $\|M - B\|$, For simplicity, we replace $\boldsymbol{x}_i$ by $\boldsymbol{x}$ and $\mathcal{E}_i$ by $\mathcal{E}$.

$$\begin{aligned}
&\|M - B\| \\
=& \|\mathbb{E}[\![B_i\mathbf{1}(\mathcal{E}_i^c)]\!]\| \\
=& \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})(\boldsymbol{b}^T\boldsymbol{x})(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})\mathbf{1}(\mathcal{E}^c)]\!] \\
\overset{\zeta_1}{\leq}& \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2(\boldsymbol{b}^T\boldsymbol{x})^2(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
=& \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, \mathbb{E}[\![(\boldsymbol{b}^T\boldsymbol{x})^2(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})^2\boldsymbol{x}\boldsymbol{x}^T]\!]\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_2}{\leq}& \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, C_{2K+3}\|\boldsymbol{\beta}\|^2\|\boldsymbol{\gamma}\|^2\Pi_{k=1}^K \operatorname{tr}(A_k)^2 I\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_3}{\leq}& \frac{\sqrt{2KC_{2K+3}}}{\sqrt{nd^P}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k)
\end{aligned} \tag{62}$$

where ζ_1 is from Holder's inequality, ζ_2 is because of Lemma 3 and ζ_3 is because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!] = \mathbb{P}[\![\mathcal{E}^c]\!]$.

According to Eq. (62) and Lemma 5, if $\frac{\sqrt{2KC_{2K+3}}}{\sqrt{nd^P}} \leq \delta/2$, then

$$\|M - B\| \leq \frac{1}{2}\delta\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) \leq \frac{1}{2}\delta\|B\| \tag{63}$$

Since $\delta \leq 1$, we also have $\|M - B\| \leq \frac{1}{2}\|B\|$, so by Lemma 5,

$$\frac{3}{2}\|B\| \geq \|M\| \geq \frac{1}{2}\|B\| \geq \frac{1}{2}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) \tag{64}$$

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Step 3. Now we bound $\|M - \hat{M}\|$. $\|M\| \leq m$ automatically holds. Since M is probably not PSD, we don't have $\|\mathbb{E}[\![M_i^2]\!]\| \leq m\|M\|$. However, we can still show that $\mathbb{E}[\![M_i^2]\!] \leq O(m)\|M\|$.

$$
\begin{aligned}
&\|\mathbb{E}[\![M_i^2]\!]\| \\
&\leq \|\mathbb{E}[\![B_i^2]\!]\| \\
&= \|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})^2\|\boldsymbol{x}\|^2\boldsymbol{x}\boldsymbol{x}^T]\!]\| \\
&\leq C_{2K+3}d \times (\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k))^2 \\
&\leq \frac{2C_{2K+3}}{(4P+5)^{K+2}}m\|M\|
\end{aligned}
\tag{65}
$$

We can use matrix Bernstein inequality now. Let $Z_i := M_i - M$. $\|Z_i\| \leq 2m$. $\|\mathbb{E}[\![Z_i^2]\!]\| \leq (\frac{2C_{2K+3}}{(4P+5)^{K+2}} + 1)m\|M\|$. Define $\hat{C}_K := \frac{2C_{2K+3}}{(4P+5)^{K+2}} + 1$, then

$$
\mathbb{P}\left[\frac{1}{n}\|\sum_{i=1}^n Z_i\| \geq \delta\|M\|\right] \leq 2d\exp(-\frac{\delta^2 n^2\|M\|^2/2}{\hat{C}_K mn\|M\| + 2mn\delta\|M\|/3}) \leq 2d\exp(-\frac{\delta^2 n\|M\|/2}{\hat{C}_K m + 2m\delta/3})
\tag{66}
$$

Thus, when $n \geq (P+1)(\frac{\hat{C}_K}{\delta^2} + \frac{2}{3\delta})m/\|M\|\log d$, we have w.p., $1 - c_2 d^{-P}$,

$$
\|\hat{M} - M\| \leq \frac{1}{3}\delta\|M\| \leq \frac{1}{2}\delta\|B\|.
$$

By Eq. (61) and Eq. (64),

$$
\begin{aligned}
(P+1)(\frac{\hat{C}_K}{\delta^2} + \frac{2}{3\delta})m/\|M\|\log d &\leq c_1\frac{\hat{C}_K}{\delta^2} \times (4P+5)^{K+3}d\log^{K+1}(n)\log(d) \\
&\leq \frac{c_1}{\delta^2}(2C_{2K+3}(P+1) + (4P+5)^{K+2})d\log^{K+1}(n)\log(d)
\end{aligned}
$$

Applying the fact, $\|\hat{B} - B\| \leq \|\hat{B} - \hat{M}\| + \|\hat{M} - M\| + \|M - B\|$, and Lemma 7 completes the proof. $\qquad\square$

A.5 Proof of Lemma 7

Proof. Assume we require $n \geq cd \log(d) \log^{K+1}(n)$ and we have $n \geq bcd \log(d) \log^A(d)$, where b, A depends only on K.

$$n \geq cd \log d \log^{K+1}(n)$$

$$\Uparrow$$

$$\frac{n}{\log^{K+1}(n)} \geq cd \log d$$

$$\Uparrow$$

$$\frac{bcd \log(d) \log^A(d)}{\log^{K+1}(bcd \log(d) \log^A(d))} \geq cd \log d$$

$$\Uparrow$$

$$b \log^A(d) \geq \log^{K+1}(bcd \log(d) \log^A(d))$$

$$\Uparrow$$

$$\log b + A \log \log(d) \geq (K+1) \log(\log(b) + \log(c) + \log(d) + (A+1) \log \log(d))$$

$$\Uparrow$$

$$\log b + A \log \log(d) \geq (K+1) \log(4 \max\{\log(b), \log(c), \log(d), (A+1) \log \log(d)\}) \tag{67}$$

$$\Uparrow$$

$$\begin{cases} \log b \geq (K+1) \log(4 \log(b)) \\ \log b \geq (K+1) \log(4 \log(c)) \\ \log b + A \log \log(d) \geq (K+1) \log(4 \log(d)) \\ \log b + A \log \log(d) \geq (K+1) \log(4(A+1) \log \log(d)) \end{cases}$$

$$\Uparrow$$

$$\begin{cases} b \geq K^{4K} \\ b \geq 4^{K+1} \log^{K+1}(c) \\ A \geq K+1 \\ b \geq (4(A+1))^{K+1} \end{cases}$$

$$\Uparrow$$

$$\begin{cases} b = K^{4K} \log^{K+1}(c) \\ A = K+1 \end{cases}$$

$$\square$$

B Proofs of Tensor Method for Initialization

B.1 Some Lemmata

We will use the following lemma to guarantee the robust tensor power method. The proofs of these lemmata will be found in Sec. B.4.

Lemma 8 (Some properties of thrid-order tensor). *If $T \in \mathbb{R}^{d \times d \times d}$ is a supersymmetric tensor, i.e.,T_{ijk} is equivalent for any permutation of the index, then the operator norm defined as*

$$\|T\|_{op} := \sup_{\|\boldsymbol{a}\|=1} |T(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a})|$$

Property 1. $\|T\|_{op} = \sup_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=\|\boldsymbol{c}\|=1} |T(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})|$

Property 2. $\|T\|_{op} \leq \|T_{(1)}\| \leq \sqrt{K} \|T\|_{op}$

Property 3. *If T is a rank-one tensor, then $\|T_{(1)}\| = \|T\|_{op}$*

Property 4. *For any matrix $W \in \mathbb{R}^{d \times d'}$, $\|T(W, W, W)\|_{op} \leq \|T\|_{op} \|W\|^3$*

Lemma 9 (Approximation error for the second moment). *Let $\{\boldsymbol{x}_i, y_i\}_{i \in [n]}$ be generated from the mixed linear regression model* (3). *Define $M_2 := \sum_{k=[K]} 2p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*$ and $\hat{M}_2 := \frac{1}{n} \sum_{i \in [n]} y_i^2 (\boldsymbol{x}_i \otimes$*

$x_i - I$). Then with $n \geq c_1 \frac{1}{p_{min}\delta_2^2} d \log^2(d)$, we have w.p. $1 - c_2 K d^{-2}$,

$$\|\hat{M}_2 - M_2\| \leq \delta_2 \sum_k p_k \|w_k^*\|^2 \tag{68}$$

where c_1, c_2 are universal constants.

And for any fixed orthogonal matrix $Y \in \mathbb{R}^{d \times K}$, with the same condition, we have

$$\|Y^T(\hat{M}_2 - M_2)Y\| \leq \delta_2 \sum_k p_k \|w_k^*\|^2 \tag{69}$$

Lemma 10 (Subspace Estimation). Let M_2, M_3 be

$$M_2 = \sum_{k=[K]} 2 p_k w_k^* \otimes w_k^*, \text{ and } M_3 = \sum_{k=[K]} 6 p_k w_k^* \otimes w_k^* \otimes w_k^*, \tag{70}$$

and $\hat{M}_2$ be an estimate of M_2. Assume $\|\hat{M}_2 - M_2\| \leq \delta \sigma_K(M_2)$ and $\delta \leq \frac{1}{6}$. Let Y be the returned matrix of the power method after $O(\log(1/\delta))$ steps. Define $R_2 = Y^T M_2 Y$ and $R_3 = M_3(Y, Y, Y)$. Then $\|R_2\| \leq \|M_2\|$ and $\|R_3\|_{op} \leq \|M_3\|_{op}$. We also have

$$\|YY^T w_k^* - w_k^*\| \leq 3\delta \|w_k^*\|, \forall k \tag{71}$$

and

$$\sigma_K(R_2) \geq \frac{3}{4} \sigma_K(M_2)$$

Lemma 11 (Approximation error for the third moment). Let $\{x_i, y_i\}_{i \in [n]}$ be drawn from the mixed linear regression model (3). Let $Y \in \mathbb{R}^{d \times K}$ be any fixed orthogonal matrix that satisfies, $\|YY^T w_k^* - w_k^*\| \leq \frac{1}{2}\|w_k^*\|, \forall k$, and $r_i = Y^T x_i$, for all $i \in [n]$. Let

$$\hat{R}_3 = \frac{1}{n} \sum_{i \in [n]} y_i^3 (r_i \otimes r_i \otimes r_i - \sum_{j \in [K]} e_j \otimes r_i \otimes e_j - \sum_{j \in [K]} e_j \otimes e_j \otimes r_i - \sum_{j \in [K]} r_i \otimes e_j \otimes e_j)$$

and

$$R_3 = \sum_{k=[K]} 6 p_k (Y^T w_k^*) \otimes (Y^T w_k^*) \otimes (Y^T w_k^*)$$

Then if $n \geq c_3 \frac{1}{p_{min}\delta_3^2} K^3 \log^4(d)$ and $3\sqrt{C_5} n^{-1/2} d^{-1} \leq \frac{\delta_3}{4}$, we have w.p. $1 - c_4 K d^{-2}$

$$\|\hat{R}_3 - R_3\|_{op} \leq \delta_3 \sum_{k \in [K]} p_k \|w_k^*\|^3,$$

where c_3 and c_4 are universal constant.

Lemma 12 (Robust Tensor Power Method. Similar to Lemma 4 in [7]). Let $R_2 = \sum_{k=1}^K p_k u_k \otimes u_k$ and $R_3 = \sum_{k=1}^K p_k u_k \otimes u_k \otimes u_k$, where $u_k \in \mathbb{R}^K$ can be any fixed vector. Define $\sigma_K := \sigma_K(R_2)$. Assume the estimations of R_2 and R_3, $\hat{R}_2$ and $\hat{R}_3$ respectively, satisfy $\|R_2 - \hat{R}_2\|_{op} \leq \epsilon_2$ and $\|R_3 - \hat{R}_3\|_{op} \leq \epsilon_3$ with

$$\epsilon_2 \leq \sigma_K/3, \ 8\|R_3\|_{op}\sigma_K^{-5/2}\epsilon_2 + 2\sqrt{2}\sigma_K^{-3/2}\epsilon_3 \leq c_T \frac{1}{K\sqrt{p_{max}}}, \tag{72}$$

for some constant c_T. Let the whitening matrix $\hat{W} = \hat{U}_2 \hat{\Lambda}_2^{-1/2} \hat{U}_2^T$, where $\hat{R}_2 = \hat{U}_2 \hat{\Lambda}_2 \hat{U}_2^T$ is the eigendecomposition of $\hat{R}_2$. Then w.p. $1 - \eta$, the eigenvalues $\{\hat{a}_k\}_{k=1}^K$ and the eigenvectors $\{\hat{v}_k\}_{k=1}^K$ computed from the whitened tensor $\hat{R}_3(\hat{W}, \hat{W}, \hat{W}) \in \mathbb{R}^{K \times K \times K}$ by using the robust tensor power method [2] will satisfy

$$\|(\hat{W}^T)^\dagger(\hat{a}_k \hat{v}_k) - u_k\| \leq \kappa_2 \epsilon_2 + \kappa_3 \epsilon_3$$

where $\kappa_2 = 3\|R_2\|^{1/2}\sigma_K^{-1} + 200\|R_2\|^{1/2}\|R_3\|_{op}\sigma_K^{-5/2}$, $\kappa_3 = 75\|R_2\|^{1/2}\sigma_K^{-3/2}$ and η is related to the computational time by $O(\log(1/\eta))$.

Remark: This lemma differs from Lemma 4 of [7] in the requirement on ϵ_2, ϵ_3. Lemma 4 in [7] treats ϵ_2, ϵ_3 in the same order (that are bounded by the same value), however, they should have different order because one is for second-order moments and the other is for third-order moments.

B.2 Proof of Theorem 3

Proof Details. We state the proof outline here,

1. $\|\hat{M}_2 - M_2\| \leq \epsilon_{M_2}$ by Matrix Bernstein's inequality.
2. $\|YY^T \boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \leq \epsilon_Y \|\boldsymbol{w}_k^*\|$ for all $k \in [K]$ by Davis-Kahan's theorem [10].
3. $\|\hat{R}_2 - R_2\| \leq \epsilon_2$ by Matrix Bernstein's inequality.
4. $\|\hat{R}_3 - R_3\|_{op} \leq \epsilon_3$ by Matrix Bernstein's inequality after matricizing tensor.
5. Let $\hat{\boldsymbol{u}}_k = (\hat{W}^T)^\dagger (\hat{a}_k \hat{\boldsymbol{v}}_k)$. Then $\|\hat{\boldsymbol{u}}_k - Y^T \boldsymbol{w}_k^*\| \leq \epsilon_{\boldsymbol{u}}$ by the robust tensor power method.
6. Finally, $\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| \leq c_6 \Delta_{min}$ by combining the results of Step 2 and Step 5.

The lemmata in Appendix B.1 provide the bound for the above steps: Lemma 9 for Step 1, Lemma 10 for Step 2 and Step 3, Lemma 11 for Step 4, and Lemma 12 for Step 5. Now we show the details. Define

$$\bar{\kappa}_2 := 4\|M_2\|^{1/2}\sigma_K^{-1}(M_2) + 412\|M_2\|^{1/2}\|M_3\|_{op}\sigma_K^{-5/2}(M_2)$$

and

$$\bar{\kappa}_3 := 116\|M_2\|^{1/2}\sigma_K^{-3/2}(M_2).$$

By Lemma 10, we have $\bar{\kappa}_3 \geq \kappa_3$ and $\bar{\kappa}_2 \geq \kappa_2$ for any orthogonal matrix Y.

$$
\begin{aligned}
\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| &\overset{\xi_1}{\leq} \|Y\hat{\boldsymbol{u}}_k - YY^T\boldsymbol{w}_k^*\| + \|YY^T\boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \\
&\overset{\xi_2}{\leq} \bar{\kappa}_2\|\hat{R}_2 - R_2\| + \bar{\kappa}_3\|\hat{R}_3 - R_3\|_{op} + \frac{2}{3}\delta_{M_2}\|\boldsymbol{w}_k^*\|\sigma_K^{-1}(M_2)\sum_k p_k\|\boldsymbol{w}_k^*\|^2 \\
&\overset{\xi_3}{\leq} \bar{\kappa}_2\delta_2\sum_k p_k\|\boldsymbol{w}_k^*\|^2 + \bar{\kappa}_3\delta_3\sum_k p_k\|\boldsymbol{w}_k^*\|^3 + \frac{2}{3}\delta_{M_2}\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2
\end{aligned}
$$

$$\tag{73}$$

where ξ_1 is due to triangle inequality, ξ_2 is due to Lemma 12, Lemma 10 and Lemma 9, and ξ_3 is due to Lemma 9 and Lemma 11. Therefore, we can set

$$\delta_2 \leq \frac{c_6\Delta_{min}}{3\bar{\kappa}_2\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

$$\delta_3 \leq \frac{c_6\Delta_{min}}{3\bar{\kappa}_3\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^3}$$

and

$$\delta_{M_2} \leq \frac{c_6\Delta_{min}}{2\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

such that $\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| \leq c_6\Delta_{min}$. Note that Lemma 12 also requires Eq. (72), which can be satisfied if

$$\|\hat{R}_2 - R_2\| \leq \min\{\frac{\sigma_K(M_2)}{4}, \frac{c_T\sigma_K(M_2)^{5/2}}{34\|M_3\|_{op}K\sqrt{p_{max}}}\}$$

and

$$\|\hat{R}_3 - R_3\|_{op} \leq \frac{c_T\sigma_K(M_2)^{3/2}}{6K\sqrt{p_{max}}}.$$

Therefore, we require

$$\delta_2 \leq \delta_2^* := \frac{1}{\sum_k p_k\|\boldsymbol{w}_k^*\|^2}\min\{\frac{\sigma_K(M_2)}{4}, \frac{c_T\sigma_K(M_2)^{5/2}}{34\|M_3\|_{op}K\sqrt{p_{max}}}, \frac{c_6\Delta_{min}}{3\bar{\kappa}_2}\}$$

$$\delta_3 \leq \delta_3^* := \frac{1}{\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^3}\min\{\frac{c_6\Delta_{min}}{3\bar{\kappa}_3}, \frac{c_T\sigma_K(M_2)^{3/2}}{6K\sqrt{p_{max}}}\}$$

$$\delta_{M_2} \leq \delta_{M_2}^* := \frac{c_6\Delta_{min}}{2\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

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Now we analyze the sample complexity. $\delta^*_{M_2}, \delta^*_2, \delta^*_3$ correspond to the sample sets, Ω_{M_2}, Ω_2 and Ω_3 respectively. By Lemma 9, Lemma 11, we require

$$|\Omega_{M_2}| \geq c_{M_2} \frac{1}{p_{min}\delta^{*2}_{M_2}} d\log^2(d)$$

$$|\Omega_2| \geq c_2 \frac{1}{p_{min}\delta^{*2}_2} d\log^2(d)$$

$$|\Omega_3| \geq c_3 \frac{1}{p_{min}\delta^{*2}_3} K^3 \log^{11/2}(d),$$

and $3\sqrt{C_5}n^{-1/2}d^{-1} \leq \frac{\delta_3}{4}$. For the probability, we can set $\eta = d^{-2}$ in Lemma 12 by scarifying a little more computational time, which is in the order of $O(\log(d))$. Therefore, the final probability is at least $1 - O(Kd^{-2})$.

B.3 Proof of Theorem 4

According to Theorem 2, after $T_0 = O(\log d)$ iterations, we arrive the local convexity region in Corollary 1. Then we just need one more set of samples, but still need $O(\log(1/\epsilon))$ iterations to achieve $1/\epsilon$ precision. By Theorem 1, Corollary 1, Theorem 2 and Theorem 3, we can partition the dataset into $|\Omega^{(t)}| = O(d(K\log(d))^{2K+2})$ for all $t = 0, 1, 2, \cdots, T_0 + 1$ to satisfy their sample complexity requirement. This complete the proof.

B.4 Proofs of Some Lemmata

B.4.1 Proof of Lemma 8

Proof. **Property 1.** See the proof in Lemma 21 of [19].

Property 2.

$$\|T_{(1)}\| = \max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F \leq \max_{\|\boldsymbol{a}\|=1} \sqrt{K}\|T(\boldsymbol{a}, I, I)\| = \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \sqrt{K}|T(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b})| = \|T\|_{op}.$$

Obviously, $\max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F \geq \|T\|_{op}$.
Property 3. Let $T = \boldsymbol{v} \otimes \boldsymbol{v} \otimes \boldsymbol{v}$.

$$\|T_{(1)}\| = \max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F = \max_{\|\boldsymbol{a}\|=1} \|\boldsymbol{v}\|^2(\boldsymbol{v}^T\boldsymbol{a})^2 = \|\boldsymbol{v}\|^3 = \max_{\|\boldsymbol{a}\|=1} |(\boldsymbol{v}^T\boldsymbol{a})^3| = \|T\|_{op}.$$

Property 4. There exists a $\boldsymbol{u} \in \mathbb{R}^{d'}$ with $\|\boldsymbol{u}\| = 1$ such that

$$\|T(W, W, W)\|_{op} = |T(W\boldsymbol{u}, W\boldsymbol{u}, W\boldsymbol{u})| \leq \|T\|_{op}\|W\boldsymbol{u}\|^3 \leq \|T\|_{op}\|W\|^3$$

$\square$

B.4.2 Proof of Lemma 9

Proof. Define $M_2^{(k)} := 2\boldsymbol{w}_k^*\boldsymbol{w}_k^{*T}$ and $\hat{M}_2^{(k)} = \frac{1}{|S_k|}\sum_{i \in S_k} y_i^2(\boldsymbol{x}_i \otimes \boldsymbol{x}_i - I)$, where $S_k \subset [n]$ is the index set for samples from the k-th model. Since we assume $|S_k| = p_k n$, $\hat{M}_2 = \sum_{k \in [K]} p_k M_2^{(k)}$. We first bound $\|\hat{M}_2^{(k)} - M_2^{(k)}\|$. By Lemma 4 with $K = 1$, $A_1 = \boldsymbol{w}_k^*\boldsymbol{w}_k^{*T}$, then if $|S_k| \geq c_1 \frac{1}{\delta^2} d\log^2(d)$, we have w.p., $1 - c_2 d^{-2}$,

$$\left\|\frac{1}{|S_k|}\sum_{i \in S_k} y_i^2 \boldsymbol{x}_i\boldsymbol{x}_i^T - \|\boldsymbol{w}_k^*\|^2 I - 2\boldsymbol{w}_k^*\boldsymbol{w}_k^*\right\| \leq \delta\|\boldsymbol{w}_k^*\|^2.$$

By Lemma 4 with $K = 0$, we have w.p. at least $1 - d^{-2}$,

$$\left\|\frac{1}{|S_k|}\sum_{i \in S_k} \boldsymbol{x}_i\boldsymbol{x}_i^T - I\right\| \leq \delta$$

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Then
$$\|\frac{1}{|S_k|}\sum_{i\in S_k}(\boldsymbol{x}_i^T\boldsymbol{w}_k^*)^2 - \|\boldsymbol{w}_k^*\|^2\| \leq \|\frac{1}{|S_k|}\sum_{i\in S_k}\boldsymbol{x}_i\boldsymbol{x}_i^T - I\|\|\boldsymbol{w}_k^*\|^2 \leq \delta\|\boldsymbol{w}_k^*\|^2.$$

Thus
$$\|\frac{1}{|S_k|}\sum_{i\in S_k}y_i^2(\boldsymbol{x}_i\boldsymbol{x}_i^T - I) - 2\boldsymbol{w}_k^*\boldsymbol{w}_k^*\| \leq 2\delta\|\boldsymbol{w}_k^*\|^2.$$

And w.p. $1 - O(Kd^{-2})$,
$$\|\hat{M}_2 - M_2\| \leq 2\delta\sum_k p_k\|\boldsymbol{w}_k^*\|^2.$$

$\square$

B.4.3 Proof of Lemma 10

Proof. $\|R_2\| \leq \|Y\|^2\|M_2\| = \|M_2\|$. By Property 4 in Lemma 8, $\|R_3\|_{op} \leq \|Y\|^3\|M_3\|_{op} = \|M_3\|_{op}$. Let U be the top-K eigenvectors of M_2. Then $U = \text{span}(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, \cdots, \boldsymbol{w}_K^*)$. Let $\bar{Y} \in \mathbb{R}^{d\times K}$ be the top-K eigenvectors of $\hat{M}_2$. By Lemma 9 in [19] (Davis-Kahan's theorem [10] can also prove it),
$$\|(I - \bar{Y}\bar{Y}^T)UU^T\| \leq \frac{3}{2}\delta.$$

According to Theorem 7.2 in [3], after t steps of the power method, we have

$$\|\bar{Y}\bar{Y}^T - Y^{(t)}Y^{(t)T}\| \leq (\frac{\sigma_{K+1}(\hat{M}_2)}{\sigma_K(\hat{M}_2)})^t\|\bar{Y}\bar{Y}^T - Y^{(0)}Y^{(0)T}\|.$$

When $\delta \leq 1/3$, by Weyl's inequality, we have $\sigma_{K+1}(\hat{M}_2) \leq \frac{1}{3}\sigma_K(M_2)$ and $\sigma_K(\hat{M}_2) \geq \frac{2}{3}\sigma_K(M_2)$. Therefore, after $t = \log(2/(3\delta))$ steps of the power method, we have

$$\|\bar{Y}\bar{Y}^T - Y^{(t)}Y^{(t)T}\| \leq \frac{3}{2}\delta$$

Let $Y = Y^{(t)}$. We have
$$\|YY^T - UU^T\| \leq \|YY^T - \bar{Y}\bar{Y}^T\| + \|UU^T - \bar{Y}\bar{Y}^T\| \leq 3\delta$$

and
$$\|YY^T\boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \leq \|YY^T - UU^T\|\|\boldsymbol{w}_k^*\| \leq 3\delta\|\boldsymbol{w}_k^*\|$$

Now we consider $\sigma_K(R_2)$. The proof is similar to that for Property 3 in Lemma 9 in [19].

$$\sigma_K(R_2) \geq \sigma_K(M_2)\sigma_K^2(Y^TU)$$

Note that $\|Y_\perp^TU\| = \|YY^T - UU^T\|$, where $Y_\perp$ is the subspace orthogonal to Y. For any normalized vector $\boldsymbol{v}$,
$$\|Y^TU\boldsymbol{v}\|^2 = \|U\boldsymbol{v}\|^2 - \|Y_\perp^TU\boldsymbol{v}\|^2 \geq 1 - (3\delta)^2 \geq \frac{3}{4}$$

Therefore, we have $\sigma_K(R_2) \geq \frac{3}{4}\sigma_K(M_2)$. $\square$

B.4.4 Proof of Lemma 11

Proof. We prove it by matricizing the tensor. Define

$$G_i = y_i^3(\boldsymbol{r}_i\otimes\boldsymbol{r}_i\otimes\boldsymbol{r}_i - \sum_{j\in[K]}\boldsymbol{e}_j\otimes\boldsymbol{r}_i\otimes\boldsymbol{e}_j - \sum_{j\in[K]}\boldsymbol{e}_j\otimes\boldsymbol{e}_j\otimes\boldsymbol{r}_i - \sum_{j\in[K]}\boldsymbol{r}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_j).$$

Like in Lemma 9, we first bound $\|\hat{R}_3^{(k)} - R_3^{(k)}\|_{op}$, where $\hat{R}_3^{(k)} = \frac{1}{|S_k|}\sum_{i\in S_k}G_i$, and $R_3^{(k)} = 6(Y^T\boldsymbol{w}_k^*)\otimes(Y^T\boldsymbol{w}_k^*)\otimes(Y^T\boldsymbol{w}_k^*)$.

$$\|R_3^{(k)}\|_{op} = 6\|Y^T\boldsymbol{w}_k^*\|^3.$$

By Lemma 10, $\frac{1}{2}\|\boldsymbol{w}_k^*\| \leq \|Y^T\boldsymbol{w}_k^*\| \leq \frac{3}{2}\|\boldsymbol{w}_k^*\|$. Thus

$$\frac{3}{4}\|\boldsymbol{w}_k^*\|^3 \leq \|R_3^{(k)}\|_{op} \leq \frac{81}{4}\|\boldsymbol{w}_k^*\|^3. \tag{74}$$

Then

$$\|G_i\|_{op} \leq 4|\boldsymbol{x}_i^T\boldsymbol{w}_k^*|^3\|\boldsymbol{r}_i\|^3, \tag{75}$$

By Corollary 4, we have w.p., $1 - n^{-1}d^{-2}$, $\|\boldsymbol{r}_i\|^2 \leq 4K\log n$. Thus, w.p. $1 - 4n^{-1}d^{-2}$,

$$\|G_i\|_{op} \leq 4 \times 12^{3/2}\|\boldsymbol{w}_k^*\|^3\log^3(n)(4K)^{3/2}$$

Define $m := c_6\|\boldsymbol{w}_k^*\|^3 K^{3/2}\log^3(n)$ for constant $c_6 = 4 \times (48)^{3/2}$, and the event

$$\mathcal{E}_i := \{\|G_i\|_{op} \leq m\}$$

Then $\mathbb{P}[\mathcal{E}_i^c] \leq 4n^{-1}d^{-2}$. Define a new tensor $B_i = \mathbf{1}(\mathcal{E}_i)G_i$, its expectation $B = \mathbb{E}[B_i]$ (the expectation is over all samples from the k-th components) and its empirical average $\hat{B} = \frac{1}{|S_k|}\sum_{i\in[S_k]}B_i$.

Step 1. So we have $B_i = G_i$ for all $i \in S_k$ w.p. $1 - 4d^{-2}$, i.e.,

$$\hat{R}_3^{(k)} = \hat{B} \tag{76}$$

Step 2. We bound $\|B - R_3^{(k)}\|_{op}$

$$\begin{aligned}
\|B - R_3^{(k)}\|_{op} &= \|\mathbb{E}[\mathbf{1}(\mathcal{E}_i^c)G_i]\|_{op} \\
&= \max_{\|\boldsymbol{a}\|=1}|\mathbb{E}[\mathbf{1}(\mathcal{E}_i^c)G_i(\boldsymbol{a},\boldsymbol{a},\boldsymbol{a})]| \\
&\leq \mathbb{E}[\mathbf{1}(\mathcal{E}_i^c)]^{1/2}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}[G_i(\boldsymbol{a},\boldsymbol{a},\boldsymbol{a})^2]|^{1/2} \\
&\leq 2n^{-1/2}d^{-1}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}[(y_i^3((\boldsymbol{r}_i^T\boldsymbol{a})^3 - 3\boldsymbol{r}_i^T\boldsymbol{a}))^2]|^{1/2} \\
&\leq 2n^{-1/2}d^{-1}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x}_i)^6((\boldsymbol{x}_i^TY\boldsymbol{a})^6 + 9(\boldsymbol{x}_i^TY\boldsymbol{a})^2)]|^{1/2} \\
&\leq 2n^{-1/2}d^{-1}\sqrt{2C_5}\|\boldsymbol{w}_k^*\|^3 \\
&\overset{\xi}{\leq} 3\sqrt{C_5}n^{-1/2}d^{-1}\|R_3^{(k)}\|_{op},
\end{aligned} \tag{77}$$

where ξ is due to Eq. (74). Therefore, if $3\sqrt{C_5}n^{-1/2}d^{-1} \leq \frac{\delta_3}{4}$, we have

$$\|B - R_3^{(k)}\|_{op} \leq \frac{3\delta_3}{8}\|\boldsymbol{w}_k^*\|^3 \leq \frac{\delta_3}{2}\|R_3^{(k)}\|_{op} \tag{78}$$

And further if $\delta_3 \leq 1$, combining Eq. (74),

$$\frac{3}{8}\|\boldsymbol{w}_k^*\|^3 \leq \frac{1}{2}\|R_3^{(k)}\|_{op} \leq \|B\|_{op} \leq \frac{3}{2}\|R_3^{(k)}\|_{op} \leq 32\|\boldsymbol{w}_k^*\|^3$$

Step 3. We bound $\|\hat{B} - B\|_{op}$. Let $Z_i = (B_i - B)_{(1)}$.

$$\begin{aligned}
\|B_{(1)}\| &\leq \max_{\|\boldsymbol{a}\|=1}\|B_{(1)}\boldsymbol{a}\| \\
&= \max_{\|\boldsymbol{a}\|=1}\|B(\boldsymbol{a},I,I)\|_F \\
&\leq \max_{\|\boldsymbol{a}\|=1}K\|B(\boldsymbol{a},I,I)\| \\
&\leq \max_{\|\boldsymbol{a}\|=1}\max_{\|\boldsymbol{b}\|=1}\sqrt{K}|B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})| \\
&\overset{\xi}{=} \sqrt{K}\|B\|_{op} \\
&\leq 32\sqrt{K}\|\boldsymbol{w}_k^*\|^3
\end{aligned} \tag{79}$$

where ξ is due to Lemma 8.

$$\|Z_i\| \leq \|B_{i(1)}\| + \|B_{(1)}\| \leq \sqrt{K}(\|B_i\|_{op} + \|B\|_{op}) \leq 2\sqrt{K}m$$

Now consider $\|\mathbb{E}[Z_i Z_i^T]\|$ and $\|\mathbb{E}[Z_i^T Z_i]\|$.

$$\mathbb{E}[Z_i Z_i^T] = \mathbb{E}[(B_{i(1)} - B_{(1)})(B_{i(1)} - B_{(1)})^T] = \mathbb{E}[B_{i(1)} B_{i(1)}^T] - B_{(1)} B_{(1)}^T$$

$$\begin{aligned}
\|\mathbb{E}[B_{i(1)} B_{i(1)}^T]\| &\leq \|\mathbb{E}[G_{i(1)} G_{i(1)}^T]\| \\
&\leq \|\mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^4 \boldsymbol{r}\boldsymbol{r}^T + 2\|\boldsymbol{r}\|^2 I + (K+6)\boldsymbol{r}\boldsymbol{r}^T - 6\|\boldsymbol{r}\|^2\boldsymbol{r}\boldsymbol{r}^T)]\| \\
&\leq \|\mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|Y^T\boldsymbol{x}\|^4 Y^T\boldsymbol{x}\boldsymbol{x}^T Y + 2\|Y^T\boldsymbol{x}\|^2 I + (K+6)Y^T\boldsymbol{x}\boldsymbol{x}^T Y)]\| \\
&\leq 2C_5 K^2 \|\boldsymbol{w}_k^*\|^6,
\end{aligned} \tag{80}$$

where the last inequality is due to Lemma 3. Thus

$$\|\mathbb{E}[Z_i Z_i^T]\| \leq 3C_5 K^2 \|\boldsymbol{w}_k^*\|^6$$

Similarly $\mathbb{E}[Z_i^T Z_i] = \mathbb{E}[B_{i(1)}^T B_{i(1)}] - B_{(1)}^T B_{(1)}$ and $\|B_{(1)}^T B_{(1)}\| \leq \|B_{(1)}\|^2$.

$$\begin{aligned}
&\|\mathbb{E}[B_{i(1)}^T B_{i(1)}]\| \\
&\leq \|\mathbb{E}[G_{i(1)}^T G_{i(1)}]\| \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[y_i^6 \|\boldsymbol{r}^T A\boldsymbol{r}\boldsymbol{r} - (2A\boldsymbol{r} + \text{tr}(A)\boldsymbol{r})\|^2] \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6((\boldsymbol{r}^T A\boldsymbol{r})^2\|\boldsymbol{r}\|^2 + 4\boldsymbol{r}^T A^2\boldsymbol{r} + \text{tr}^2(A)\|\boldsymbol{r}\|^2 + |\text{tr}(A)\boldsymbol{r}^T A\boldsymbol{r}|(4 + 2\|\boldsymbol{r}\|^2))] \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^6\|A\|_F^2 + 4\|\boldsymbol{r}\|^2\text{tr}(A^2) + \text{tr}^2(A)\|\boldsymbol{r}\|^2 + (4 + 2\|\boldsymbol{r}\|^2)\|\boldsymbol{r}\|^2|\text{tr}(A)|\|A\|_F)] \\
&\leq \mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^6 + 4\|\boldsymbol{r}\|^2 + K\|\boldsymbol{r}\|^2 + \sqrt{K}(4 + 2\|\boldsymbol{r}\|^2)\|\boldsymbol{r}\|^2)] \\
&= \mathbb{E}[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|Y^T\boldsymbol{x}\|^6 + 2\sqrt{K}\|Y^T\boldsymbol{x}\|^4 + 4(\sqrt{K} + 1)\|Y^T\boldsymbol{x}\|^2 + K\|Y^T\boldsymbol{x}\|^2)] \\
&\leq 2C_5 K^3 \|\boldsymbol{w}_k^*\|^6
\end{aligned} \tag{81}$$

Therefore,

$$\|\mathbb{E}[Z_i^T Z_i]\| \leq 3C_5 K^3 \|\boldsymbol{w}_k^*\|^6,$$

and

$$\max\{\|\mathbb{E}[Z_i^T Z_i]\|, \|\mathbb{E}[Z_i Z_i^T]\|\} \leq 3C_5 K^3 \|\boldsymbol{w}_k^*\|^6 \leq c_{m2} K^{3/2} m\|\boldsymbol{w}_k^*\|^3$$

Now we are ready to apply matrix Bernstein's inequality.

$$\mathbb{P}\left[\frac{1}{|S_k|}\|\sum_{i \in S_k} Z_i\| \geq t\right] \leq 2K^2 \exp\left(-\frac{-|S_k|t^2/2}{c_{m2}K^{3/2}m\|\boldsymbol{w}_k^*\|^3 + 2\sqrt{K}mt/3}\right) \tag{82}$$

Setting $t = \delta_3 \|\boldsymbol{w}_k^*\|^3$, we have when

$$|S_k| \geq \hat{c}_3 \frac{1}{\delta_3^2} K^3 \log^3(n)\log(d) \tag{83}$$

w.p. $1 - d^{-2}$,

$$\|\hat{B} - B\|_{op} \leq \|\frac{1}{|S_k|}\sum_{i \in S_k} Z_i\| \leq \delta_3 \|\boldsymbol{w}_k^*\|^3, \tag{84}$$

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for some universal constant $\hat{c}_3$. And there exists some constant c_3, such that $|S_k| \geq \hat{c}_3 \frac{1}{\delta_3^2} K^3 \log^4(d)$ will imply (83). **Step 4.** Combing all the K components. With above three steps for k-th component, i.e., Eq. (76), Eq. (78) and Eq. (84), w.h.p., we have

$$\|\hat{R}_3^{(k)} - R_3^{(k)}\|_{op} \leq \delta_3 \|\boldsymbol{w}_k^*\|^3$$

Now we can complete the proof by combing all the K components, w.p. $1 - O(Kd^{-2})$

$$\|\hat{R}_3 - R_3\|_{op} \leq \sum_{k \in [K]} p_k \|\hat{R}_3^{(k)} - R_3^{(k)}\|_{op} \leq \delta_3 \sum_{k \in [K]} p_k \|\boldsymbol{w}_k^*\|^3 \tag{85}$$

$\square$

B.4.5 Proof of Lemma 12

Proof. Most part of the proof follows the proof of Lemma 4 in [7]. Let $\hat{W}^T R_2 \hat{W} = U\Lambda U^T$. Define $W := \hat{W}U\Lambda^{-1/2}U^T$, then W is the whitening matrix of R_2, i.e., $W^T R_2 W = I$. Define the whitened tensor $T = R_3(W, W, W)$, i.e.,

$$
\begin{aligned}
T &:= \sum_{k=1}^{K} p_k W^T \boldsymbol{u}_k \otimes W^T \boldsymbol{u}_k \otimes W^T \boldsymbol{u}_k \\
&= \sum_{k=1}^{K} p_k^{-1/2} (p_k^{1/2} W^T \boldsymbol{u}_k) \otimes (p_k^{1/2} W^T \boldsymbol{u}_k) \otimes (p_k^{1/2} W^T \boldsymbol{u}_k) \\
&= \sum_{k=1}^{K} p_k^{-1/2} \boldsymbol{v}_k \otimes \boldsymbol{v}_k \otimes \boldsymbol{v}_k,
\end{aligned}
\tag{86}
$$

where $\{\boldsymbol{v}_k := p_k^{1/2} W^T \boldsymbol{u}_k\}_{k=1}^{K}$ are orthogonal basis because $\sum_{k=1}^{K} \boldsymbol{v}_k \boldsymbol{v}_k^T = W^T R_2 W = I_K$. In practice, we have $\hat{T} := \hat{M}_3(\hat{W}, \hat{W}, \hat{W})$, an estimation of T. Define $\epsilon_T := \|\hat{T} - T\|_{op}$. Similar to the proof of Lemma 4 in [7], we have

$$
\begin{aligned}
\epsilon_T =& \|R_3(W, W, W) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\
\leq& \|R_3(W, W, W) - R_3(W, W, \hat{W})\|_{op} + \|R_3(W, W, \hat{W}) - R_3(W, \hat{W}, \hat{W})\|_{op} \\
&+ \|R_3(W, \hat{W}, \hat{W}) - R_3(\hat{W}, \hat{W}, \hat{W})\|_{op} + \|R_3(\hat{W}, \hat{W}, \hat{W}) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\
=& \|R_3(W, W, W - \hat{W})\|_{op} + \|R_3(W, W - \hat{W}, \hat{W})\|_{op} \\
&+ \|R_3(W - \hat{W}, \hat{W}, \hat{W})\|_{op} + \|R_3(\hat{W}, \hat{W}, \hat{W}) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\
\leq& \|R_3\|_{op}(\|W\|^2 + \|W\|\|\hat{W}\| + \|\hat{W}\|^2)\epsilon_W + \|\hat{W}\|^3 \epsilon_3
\end{aligned}
\tag{87}
$$

where $\epsilon_W = \|\hat{W} - W\|$.

If $\epsilon_2 \leq \sigma_K/3$, we have $|\sigma_K(\hat{R}_2) - \sigma_K| \leq \epsilon_2 \leq \sigma_K/3$. Then $\frac{2}{3}\sigma_K \leq \sigma_K(\hat{R}_2) \leq \frac{4}{3}\sigma_K$ and $\|\hat{W}\| \leq \sqrt{2}\sigma_K^{-1/2}$.

$$\epsilon_W = \|\hat{W} - W\| = \|\hat{W}(I - U\Lambda^{-1/2}U^T)\| \leq \|\hat{W}\|\|I - \Lambda^{-1/2}\| \tag{88}$$

Since we have $\|I - \Lambda\| = \|\hat{W}^T R_2 \hat{W} - \hat{W}^T \hat{R}_2 \hat{W}\| \leq \|\hat{W}\|^2 \epsilon_2 = 2\sigma_K^{-1}\epsilon_2$. Thus

$$\|I - \Lambda^{-1/2}\| \leq \max\{|1 - (1 + 2\epsilon_2/\sigma_K)^{-1/2}|, |1 - (1 - 2\epsilon_2/\sigma_K)^{-1/2}|\} \leq \epsilon_2/\sigma_K$$

Therefore,

$$\epsilon_W \leq \sqrt{2}\epsilon_2 \sigma_K^{-3/2} \tag{89}$$

Now we have

$$\epsilon_T \leq 8\|R_3\|_{op}\sigma_K^{-5/2}\epsilon_2 + 2\sqrt{2}\sigma_K^{-3/2}\epsilon_3 \tag{90}$$

Thus we can apply Theorem 5.1 [2] to show the guarantees of the robust tensor power method to recover $\{\boldsymbol{v}_k\}_{k=1}^{K}$ and $\{p_k\}_{k=1}^{K}$. It can be stated as below, for some universal constant c_T and a small

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value η (the computational complexity is related to η by $O(\log(1/\eta))$), if $\epsilon_T \leq c_T \frac{1}{K\sqrt{p_{max}}}$, w.p. $1 - \eta$ the returned eigenvectors $\{\hat{\boldsymbol{v}}_k\}_{k=1}^K$ and eigenvalues $\{\hat{a}_k\}_{k=1}^K$ satisfy

$$\|\hat{\boldsymbol{v}}_k - \boldsymbol{v}_k\| \leq 8\epsilon_T\sqrt{p_k} \leq 8\epsilon_T\sqrt{p_{max}}, \ |\hat{a}_k - \frac{1}{\sqrt{p_k}}| \leq 5\epsilon_T \tag{91}$$

Let $a_k = \frac{1}{\sqrt{p_k}}$. Now we show

$$
\begin{aligned}
\|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - \boldsymbol{u}_k\| &= \|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - W^\dagger a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - (\hat{W}^T)^\dagger(a_k\boldsymbol{v}_k)\| + \|(\hat{W}^T)^\dagger(a_k\boldsymbol{v}_k) - (W^T)^\dagger a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger\|(\|\hat{a}_k\hat{\boldsymbol{v}}_k - \hat{a}_k\boldsymbol{v}_k\| + \|\hat{a}_k\boldsymbol{v}_k - a_k\boldsymbol{v}_k\|) + \|(\hat{W}^T)^\dagger - (W^T)^\dagger\|\|a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger\|(\hat{a}_k 8\epsilon_T/a_k + 5\epsilon_T) + \|(\hat{W}^T)^\dagger - (W^T)^\dagger\|a_k
\end{aligned}
\tag{92}
$$

If $\epsilon_T \leq \frac{1}{10\sqrt{p_{max}}}$, we have $\hat{a}_k/a_k \leq 3/2$. If $\epsilon_2 \leq \sigma_K/3$,

$$\|(\hat{W}^T)^\dagger\| = \|\hat{\Lambda}_2\|^{1/2} \leq \sqrt{2}\|R_2\|^{1/2} \tag{93}$$

and

$$
\begin{aligned}
\|(\hat{W}^T)^\dagger - (W^T)^\dagger\| &= \|(\hat{W}^T)^\dagger(I - U\Lambda^{1/2}U^T)\| \\
&= \|(\hat{W}^T)^\dagger\|\|I - \Lambda^{1/2}\| \\
&\leq 2\sqrt{2}\|R_2\|^{1/2}\epsilon_2/\sigma_K
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
\|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - \boldsymbol{u}_k\| &\leq \|R_2\|^{1/2}(25\epsilon_T + 3\epsilon_2/\sigma_K) \\
&\leq (3\|R_2\|^{1/2}\sigma_K^{-1} + 200\|R_2\|^{1/2}\|R_3\|_{op}\sigma_K^{-5/2})\epsilon_2 + (75\|R_2\|^{1/2}\sigma_K^{-3/2})\epsilon_3
\end{aligned}
\tag{95}
$$

$\square$

C Proofs of Subspace Clustering

C.1 Some Properties of the Distance between Subspaces

According to [16], $D(U,V) = \sqrt{r - \|U^TV\|_F^2} = \sqrt{\text{tr}(I_r - U^TVV^TU)} = \|U_\perp^TV\|_F = \|V_\perp^TU\|_F$. We briefly give the proof.

$$\|UU^T - VV^T\|_F^2 = \|(I - VV^T)UU^T - VV^T(I - UU^T)\|_F^2$$

Since $(I - VV^T)UU^T(VV^T(I - UU^T))^T = 0$ and $(VV^T(I - UU^T))^T(I - VV^T)UU^T = 0$, we have

$$
\begin{aligned}
&\|(I - VV^T)UU^T - VV^T(I - UU^T)\|_F^2 \\
=&\|(I - VV^T)UU^T\|_F^2 + \|VV^T(I - UU^T)\|_F^2 \\
=&2\,\text{tr}(VV^T - V^TUU^TV) \\
=&2(r - \|V^TU\|_F^2)
\end{aligned}
\tag{96}
$$

By the property of Frobineous norm we see $D(\cdot,\cdot)$ is a metric, so we can use triangular inequality. We will also use the following inequality, which is due to the dual property of matrices, ref. Lemma 3.2 in [5]. Let A, B be two matrices.

$$
\begin{aligned}
&\|AB\|_F \\
&= \langle AB, AB \rangle^{1/2} \\
&= \langle BB^T, A^TA \rangle^{1/2} \\
&\leq \|BB^T\|^{1/2}\|A^TA\|_*^{1/2} \\
&= \|B\|\|A\|_F
\end{aligned}
\tag{97}
$$

Similarly, we also have $\|AB\|_F \leq \|A\|\|B\|_F$.

C.2 Proof of Theorem 5

Proof. For simplicity, we use U_j to denote U_j^t for $j \in [K]$. Consider fixing $\{U_k\}_{k \neq j}$ and updating U_j.

$$
\begin{aligned}
\bar{U}_j &= \sum_{i=1}^{N} \big(\Pi_{k \neq j} \langle I_d - U_k U_k^T, \boldsymbol{z}_i \boldsymbol{z}_i^T \rangle\big) \boldsymbol{z}_i \boldsymbol{z}_i^T U_j \\
&= \sum_{q=1}^{K} \sum_{i \in \Omega_q^{(t)}} \big(\Pi_{k \neq j} \langle I_d - U_k U_k^T, U_q^* \boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T} \rangle\big) U_q^* \boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T} U_j \\
&= \sum_{q=1}^{K} U_q^* \sum_{i \in \Omega_q^{(t)}} \big(\Pi_{k \neq j} \langle I_r - U_q^{*T} U_k U_k^T U_q^*, \boldsymbol{s}_i \boldsymbol{s}_i^T \rangle\big) \boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T} U_j
\end{aligned}
\tag{98}
$$

where $\Omega_q^{(t)}$ is the set of data points belongs to q-th subspace in t-th iteration.

Define

$$
B_{jq} := \mathbb{E}\big[\Pi_{k \neq j}(\boldsymbol{s}^T(I_r - U_q^{*T} U_k U_k^T U_q^*)\boldsymbol{s})\boldsymbol{s}\boldsymbol{s}^T\big]
\tag{99}
$$

$$
\hat{B}_{jq} := \frac{1}{|\Omega_q^{(t)}|} \sum_{i = \Omega_q^{(t)}} \Pi_{k \neq j}(\boldsymbol{s}_i^T(I_r - U_q^{*T} U_k U_k^T U_q^*)\boldsymbol{s}_i)\boldsymbol{s}_i \boldsymbol{s}_i^T
\tag{100}
$$

According to Lemma 3, we have

$$
\Pi_{k \neq j} \operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) I \preceq B_{jq} \preceq C_{K-1} \Pi_{k \neq j} \operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) I
\tag{101}
$$

Note that $\operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) = D(U_q^*, U_k)^2$. We have

$$
\Pi_{k \neq j} D(U_q^*, U_k)^2 I \preceq B_{jq} \preceq C_{K-1} \Pi_{k \neq j} D(U_q^*, U_k)^2 I
\tag{102}
$$

If the conditions about n and r in Theorem 5 are satisfied, because of Lemma 4 with $\delta = \frac{1}{2C_{K-1}}$, we have, w.p. $1 - O(Kr^{-2})$,

$$
\|B_{jq} - \hat{B}_{jq}\| \leq \frac{1}{2C_{K-1}} \|B_{jq}\| \leq \frac{1}{2} \Pi_{k \neq j} \operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) \leq \frac{1}{2} \sigma_{min}(B_{jq})
$$

Therefore,

$$
\frac{1}{2} \Pi_{k \neq j} \operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) I \preceq \hat{B}_{jq} \preceq (C_{K-1} + 1) \Pi_{k \neq j} \operatorname{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) I
\tag{103}
$$

Given the condition, $D(U_k^*, U_k) \leq c_s \min_{q \neq j}\{D(U_q^*, U_j^*)\}$, we have for $q \neq k$

$$
D(U_q^*, U_k)^2 \leq (D(U_q^*, U_k^*) + D(U_k^*, U_k))^2 \leq (1 + c_s)^2 D(U_q^*, U_k^*)^2
$$

Similarly,

$$
D(U_q^*, U_k)^2 \geq (1 - c_s)^2 D(U_q^*, U_k^*)^2
$$

Therefore, for $j \neq q$

$$
\|\hat{B}_{jq}\| \leq (C_{K-1} + 1)\big(\Pi_{k:k \neq j, k \neq q}(1 + c_s)^2 D(U_q^*, U_k^*)^2\big) D(U_q^*, U_q)^2
\tag{104}
$$

For $j = q$

$$
\sigma_{min}(\hat{B}_{jj}) \geq \frac{1}{2} \Pi_{k:k \neq j}(1 - c_s)^2 D(U_j^*, U_k^*)^2
\tag{105}
$$

Shown in Eq. (98), $\bar{U}_j = \sum_{q=1}^{K} U_q^* p_q \hat{B}_{jq} U_q^{*T} U_j$. Let $[U_j^+, \bar{R}_j] := \text{QR}(\bar{U}_j)$

$$D(U_j^+, U_j^*)$$
$$= \|U_{j\perp}^{*T} \bar{U}_j \bar{R}_j^{-1}\|_F$$
$$\leq \|U_{j\perp}^{*T} \bar{U}_j\|_F \|\bar{R}_j^{-1}\|$$
$$= \|U_{j\perp}^{*T} \sum_{q=1}^{K} U_q^* p_q \hat{B}_{jq} U_q^{*T} U_j\|_F \|\bar{R}_j^{-1}\|$$
$$\leq (\sum_{q \neq j} p_q \|U_{j\perp}^{*T} U_q^*\|_F \|\hat{B}_{jq}\| \|U_q^{*T} U_j\|) \|\bar{R}_j^{-1}\|$$
$$\leq (\sum_{q \neq j} p_q \|U_{j\perp}^{*T} U_q^*\|_F \|\hat{B}_{jq}\|) \|\bar{R}_j^{-1}\|$$
$$\leq (C_{K-1}+1)(\sum_{q \neq j} p_q D(U_j^*, U_q^*)\left((1+c_s)^{2(K-2)} \Pi_{k:k\neq j, k\neq q} D(U_q^*, U_k^*)^2\right) D(U_q^*, U_q)^2) \|\bar{R}_j^{-1}\|$$
$$\leq (C_{K-1}+1)(1+c_s)^{2(K-2)} \left(\sum_{q \neq j} p_q D(U_j^*, U_q^*)\left(\Pi_{k:k\neq j, k\neq q} D(U_q^*, U_k^*)^2\right) D(U_q^*, U_q)^2\right) \|\bar{R}_j^{-1}\|$$
$$\leq (C_{K-1}+1)(1+c_s)^{2(K-2)} p_{max}(K-1) D_{max}^{2K-3} D(U_q^*, U_q)^2 \|\bar{R}_j^{-1}\|$$

$$(106)$$

Now we show,

$$\|\bar{R}_j^{-1}\| \leq \sigma_{min}^{-1}(\bar{R}_j) = \sigma_{min}^{-1}(\bar{U}_j) \leq (p_j \sigma_{min}(\hat{B}_{jj}))^{-1}$$
$$\leq \left((p_j/2)\Pi_{k:k\neq j}(1-c_s)^2 D(U_j^*, U_k^*)^2\right)^{-1}$$
$$\leq \frac{2}{p_{min}(1-c_s)^{2K-2} D_{min}^{2K-2}}$$

$$(107)$$

Combing Eq.(106), Eq. (107) and the condition on c_s,

$$D(U_j^+, U_j^*) \leq 2(C_{K-1}+1)(K-1)(1+c_s)^{2(K-2)}(1-c_s)^{-2(K-1)} \frac{p_{max} D_{max}^{2K-3}}{p_{min} D_{min}^{2K-2}} D(U_q^*, U_q)^2$$
$$\leq \frac{1}{2c_s D_{min}} \Delta_t^2$$

$$(108)$$

Using the initialization condition, we can easily obtain $\frac{\Delta_t}{2c_s D_{min}} \leq \frac{1}{2}$ by induction. Also, the condition, $D(U_j^+, U_j^*) \leq c_s D_{min}$, still holds after each update. So we have super-linear convergence rate. $\qquad\square$

D   More Experimental Results

In Fig. 2, we show that, to achieve an initial error $\epsilon^{(0)} = c$ for some constant $c < 1$, our tensor method only requires N to be proportional to d. Note that the naive initialization methods, random initialization (using normal distribution) or all-zero initialization, will lead to $\epsilon^{(0)} \approx 1.4$ and $\epsilon^{(0)} = 1$ respectively.

In Fig. 3 we compare our methods with EM in terms of iterations. In Fig. 4 we compare EM and our methods for larger K, $K = 6$. Note that the per-iteration cost of MLR will be K times more than the per-iteration cost of EM. So when K is larger, MLR will be slower than EM.

In Fig. 5, we show the sample complexities for different methods. Our methods (MLR) have a better sample complexity than EM. And the tensor initialization outperforms random initialization significantly.

Fig. 6 shows whatever the ambient dimension d is, the clusters will be exactly recovered when N is proportional to r by a constant factor.
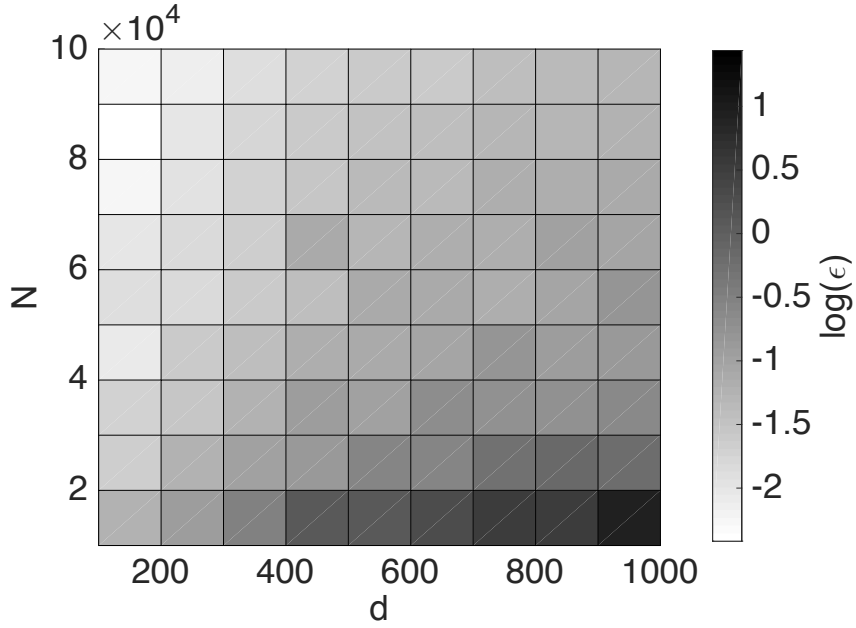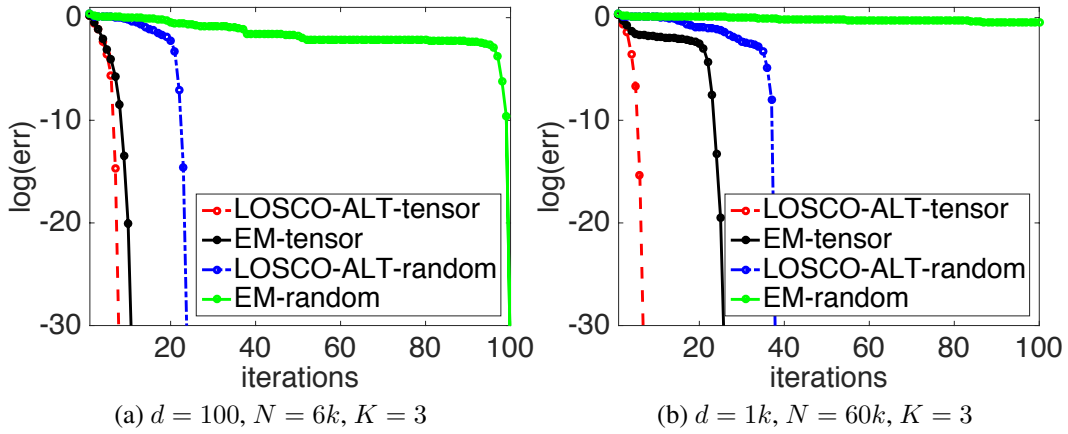
Figure 2: Initialization error for tensor method.

(a) $d = 100$, $N = 6k$, $K = 3$

(b) $d = 1k$, $N = 60k$, $K = 3$

Figure 3: Comparison with EM in terms of iterations.

Table 2: Corresponding CE's for the results in Table 1

N/K	SSC	SSC-OMP	LRR	TSC	NSN+spectral	NSN+GSR	**PSC**
200	0	0.0190	0.0010	0.0650	0	0	0
400	0	0.0090	0.0015	0.0190	0	0	0
600	0	0.0027	0	0.0120	0	0	0
800	0	0.0027	0	0.0030	0	0	0
1000	0	0.0014	0	0.0022	0	0	0

(c) $d = 60$, $N = 36k$, $K = 6$

(d) $d = 300$, $N = 180k$, $K = 6$
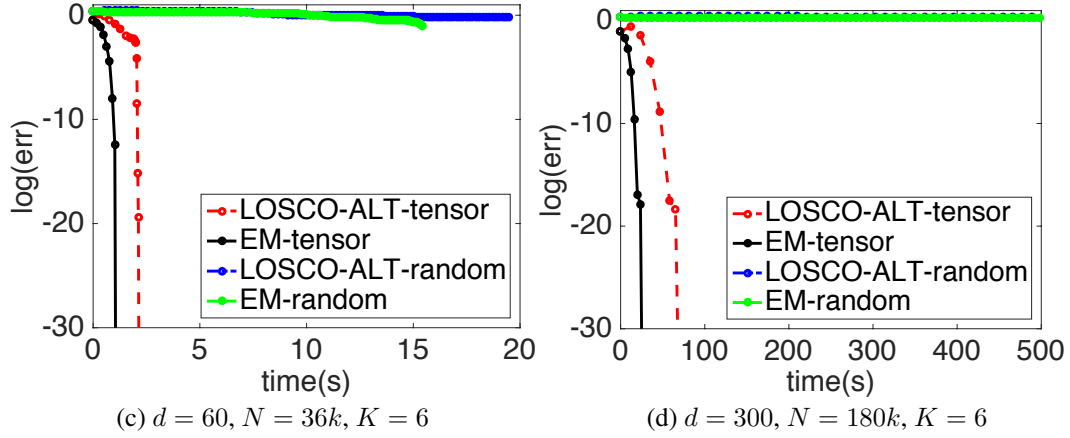
Figure 4: Comparison with EM for larger K, $K = 6$

(a) EM with random initialization

(b) EM with tensor initialization

(a) MLR with random initialization
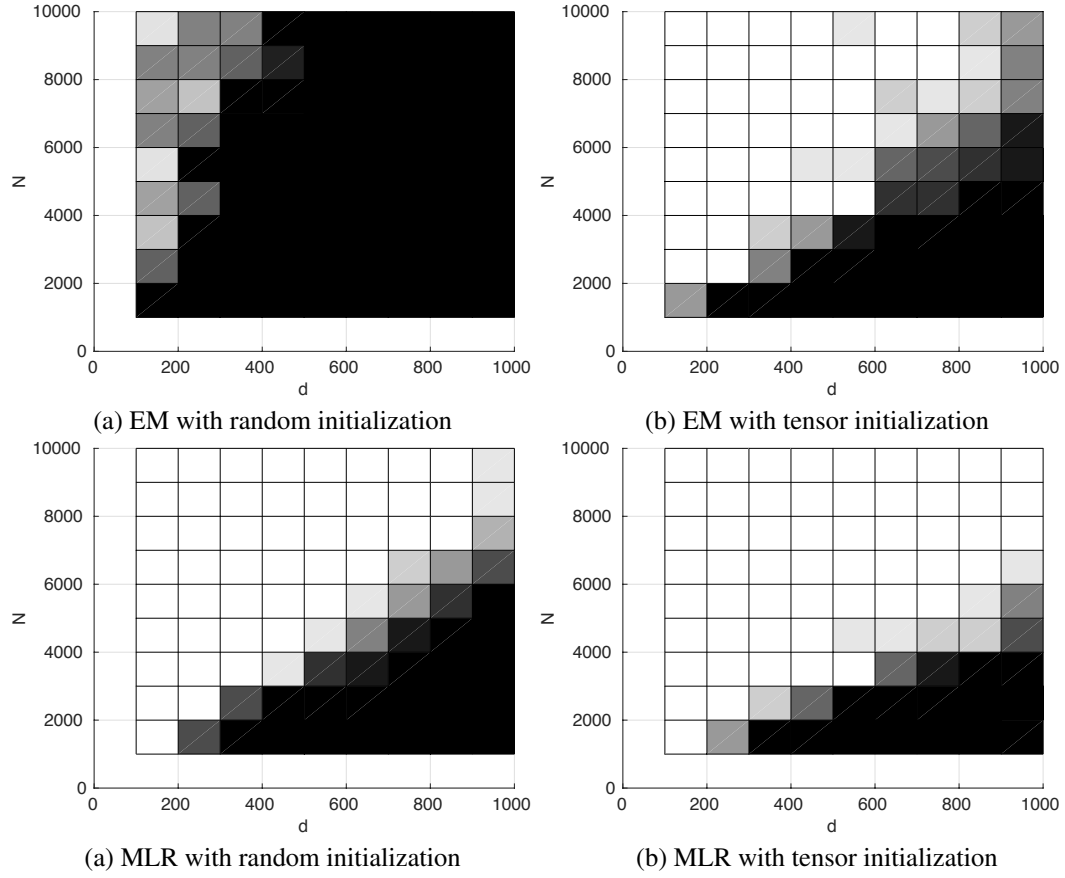
(b) MLR with tensor initialization
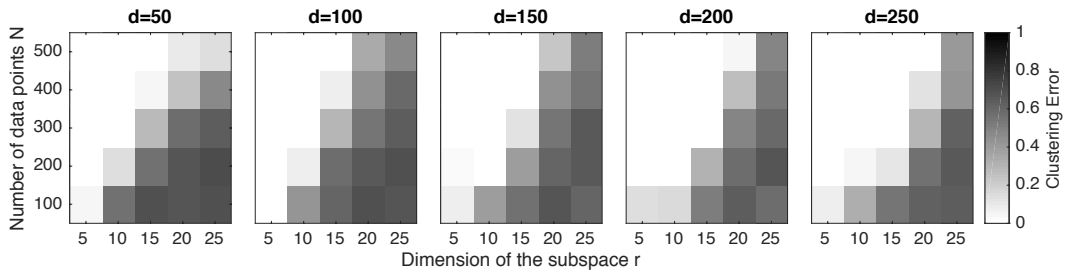
Figure 5: Sample complexities for different methods

Figure 6: Subspace Clustering error for different N, d and r.