CS243: Discrete Structures

Propositional Logic

Işıl Dillig

Course Staff

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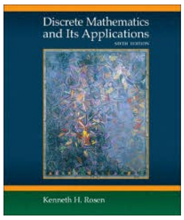
- TA office hours: Monday 4-6 PM, Tuesday 10-11 AM in McGlothin Street Hall 139
- Class meets every Tuesday, Thursday 2:00 pm - 3:20 pm

About this Course

- Give mathematical background you need for computer science
- Topics: Logic, proof techniques, number theory, probability, relations, data structures, . . .
- These will come up again and again in higher-level CS courses
  - Master CS243 material if you want to do well in future courses!

Course Resources



- Textbook: Discrete Mathematics and Its Applications by Kenneth Rosen (6th edition)
- Course webpage: http://www.cs.wm.edu/~idillig/cs243/
- Everything about the course will be posted on this page: schedule, readings from book, lecture slides, assignments, announcements, . . .

Requirements

- Weekly written homework assignments
- Two midterm exams: in-class, closed-book
  - Allowed to bring 3 pages of hand-prepared notes
- Scheduled for October 2, November 8
- Also closed-book final exam
- No make-up exams given unless you have serious, documented medical emergency

Grading

- Final exam: 40% of final grade
- Each midterm: 20% of final grade
- Homework: 20% of final grade
- Final grades may be curved, but lower bounds guaranteed (e.g., get at least A- if grade is 90% or higher)

Homework Policy

- Homework due at the beginning of class on due date
  - No late submissions accepted
  - No credit unless turned in by 2 PM on due date
  - Lowest homework score will be dropped

Homework Policy, cont.

- Homework solutions must be typeset using LaTeX
  - LaTeX markup language used in mathematical disciplines
- Course webpage has a link to LaTeX tutorial
  - Be sure to read this tutorial and get comfortable using LaTeX!
- Weilin will hold lab session on Monday to teach LaTeX
  - Room: McGlothin Street Hall 002 at 6 PM

Honor Code

- Homework write-up must be your own
- If you discuss with others, write-up must mention their names
- May not copy answers from on-line resources or other students
- Details about Honor Code in course webpage
- Honor code taken very seriously at W&M
  - May be expelled for violating honor code!

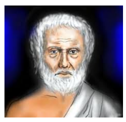
Class Participation

- Everyone expected to attend lectures
  - Key to learning material!
- Ask questions!
  - No question is a stupid question!
  - Other students also benefit from your questions
- Make class fun by participating!
  - Might win chocolate if you answer questions :)

Let's get started!

Logic

- Logic is "science of reasoning"
- Allows us to represent knowledge in precise, mathematical way
- Allows us to make valid inferences using a set of precise rules
- Roots of logic date back to the ancient Greeks, e.g., Aristotle
- Greeks were interested in valid logical inference rules, such as syllogisms
- "All men are mortal. Socrates is a men. Therefore, Socrates is mortal."



Propositional Logic

- Simplest logic is propositional logic
- Building blocks of propositional logic are propositions
- A proposition is a statement that is either true or false
- Examples:
  - "CS243 is a course in discrete mathematics": True
  - "W&M is located in California": False
  - "1+2 = 3": True

More Examples

Which of the following are propositions?

- "How are you?": No
- "It is 2 PM right now": Yes
- "x+2 = 3": No
- "Pay attention": No

Propositional Variables, Truth Value

- Truth value of a proposition identifies whether a proposition is true (written T) or false (written F)
- What is truth value of "Today is Friday"? F
- Variables that represent propositions are called propositional variables
- Denote propositional variables using lower-case letters, such as $p, p_1, p_2, q, r, s, \ldots$
- Truth value of a propositional variable is either T or F.

Compound Propositions

- Propositional logic allows constructing more complex propositions from atomic ones.
- More complex propositions formed using logical connectives (also called boolean connectives)
- Three basic logical connectives:
  1. $\wedge$: conjunction (read "and"),
  2. $\vee$: disjunction (read "or")
  3. $\neg$: negation (read "not")
- Propositions formed using these logical connectives called compound propositions; otherwise atomic propositions

Negation

- Negation of a proposition p, written $\neg p$, represents the statement "It is not the case that p".
- What is the relationship between the truth value of p and $\neg p$?
- If p is T, $\neg p$ is F and vice versa.
- In simple English, what is $\neg p$ if p stands for ...
  - "W&M is located in California"?
  - "At most 40 students are enrolled in CS243"?

Conjunction

- Conjunction of two propositions p and q, written $p \wedge q$, is the proposition "p and q"
- $p \wedge q$ is T if both p is true and q is true, and F otherwise.
- What is the conjunction and the truth value of $p \wedge q$ for ...
  - $p =$ "It is fall semester", $q =$ "Today is Thursday" ?
  - $p =$ "It is Thursday", $q =$ "It is morning" ?

Disjunction

- Disjunction of two propositions p and q, written $p \vee q$, is the proposition "p or q"
- $p \vee q$ is T if either p is true or q is true, and F otherwise.
- What is the disjunction and the truth value of $p \vee q$ for ...
  - $p =$ "It is spring semester", $q =$ "Today is Thursday"?
  - $p =$ "It is Friday", $q =$ "It is morning" ?

Propositional Formulas and Truth Tables

- A propositional formula is either an atomic or compound proposition
- Truth table for propositional formula F shows truth value of F for every possible value of its constituent atomic propositions
- Example: Truth table for $\neg p$

p	$\neg p$
T	F
F	T

- Example: Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Constructing Truth Tables

Useful strategy for constructing truth tables for a formula F:

1. Identify F's constituent atomic propositions
2. Identify F's compound propositions in increasing order of complexity, including F itself
3. Construct a table enumerating all combinations of truth values for atomic propositions
4. Fill in values of compound propositions for each row

Examples

Construct truth tables for the following formulas:

1. $(p \vee q) \wedge \neg p$
2. $(p \wedge q) \vee (\neg p \wedge \neg q)$
3. $(p \vee q \vee \neg r) \wedge r$

More Logical Connectives

- $\wedge, \vee, \neg$ most common boolean connectives, but there are other boolean connectives as well
- Other connectives: exclusive or $\oplus$, implication $\rightarrow$, biconditional $\leftrightarrow$
- Exclusive or: $p \oplus q$ is true when exactly one of p and q is true, and false otherwise
- Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (Conditional)

- An implication (or conditional) $p \rightarrow q$ is read "if p then q" or "p implies q"
- It is false if p is true and q is false, and true otherwise
- Observe: If p is false, $p \rightarrow q$ is true, regardless of q's value
- Exercise: Draw truth table for $p \rightarrow q$
- In an implication $p \rightarrow q$, p is called antecedent and q is called consequent

Converting English into Logic

Let $p =$" I major in CS" and $q =$ "I will find a good job". How do we translate following English sentences into logical formulas?

- "If I major in CS, then I will find a good job":
- "I will not find a good job unless I major in CS":
- "It is sufficient for me to major in CS to find a good job":
- "It is necessary for me to major in CS to find a good job":

More English - Logic Conversions

Let $p =$" I major in CS", $q =$ "I will find a good job", $r =$ "I can program". How do we translate following English sentences into logical formulas?

- " I will not find a good job unless I major in CS or I can program":
- " I will not find a good job unless I major in CS and I can program":
- "A prerequisite for finding a good job is that I can program":
- "If I major in CS, then I will be able to program and I can find a good job":

Converse of a Conditional

- The converse of a conditional $p \rightarrow q$ is $q \rightarrow p$.
- What is the converse of "If I major in CS, then I can program"?
- What is the converse of "If I get an A in CS243, then I am smart"?
- Question: Is it possible for a conditional to be true, but its converse to be false?

Inverse of a Conditional

- The inverse of a conditional $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- What is the inverse of "If I major in CS, then I can program"?
- What is the inverse of "If I get an A in CS243, then I am smart"?
- Question: Is it possible for a conditional to be true, but its inverse to be false?

Contrapositive of a Conditional

- The contrapositive of a conditional $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
- What is the contrapositive of "If I major in CS, then I can program"?
- What is the contrapositive of "If I get an A in CS243, then I am smart"?
- Question: Is it possible for a conditional to be true, but its contrapositive to be false?

Conditional and its Contrapositive

A conditional $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same truth value.

- Proof: We consider all four possible cases:
  - $p = T, q = T$: Both $T \rightarrow T$ and $F \rightarrow F$ are true
  - $p = T, q = f$: Both $T \rightarrow F$ and $T \rightarrow F$ are false
  - $p = F, q = T$: Both $F \rightarrow T$ and $F \rightarrow T$ are true
  - $p = F, q = F$: Both $F \rightarrow F$ and $T \rightarrow T$ are true

Question

- Consider a conditional $p \to q$
- Is it possible that its converse is true, but inverse is false?

Summary

- Conditional is of the form $p \to q$
- Converse: $q \to p$
- Inverse: $\neg p \to \neg q$
- Contrapositive: $\neg q \to \neg p$
- Conditional and contrapositive have same truth value
- Inverse and converse always have same truth value

Biconditionals

- So far, we saw five logical connectives: $\wedge, \vee, \neg, \oplus, \to$
- There is one more logical connective: biconditional, writen $\leftrightarrow$
- A biconditional $p \leftrightarrow q$ is the proposition "p if and only if q".
- The biconditional $p \leftrightarrow q$ is true if p and q have same truth value, and false otherwise.
- Exercise: Construct a truth table for $p \leftrightarrow q$
- Question: How can we express $p \leftrightarrow q$ using the other boolean connectives?

Operator Precedence

- Given a formula $p \wedge q \vee r$, do we parse this as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$?
- Without settling on a convention, formulas without explicit paranthesization are ambiguous.
- To avoid ambiguity, we will specify precedence for logical connectives.
- Operator precedence is a convention that tells us how to parse formulas if they are not explicitly paranthesized.

Operator Precedence, cont.

- Negation ($\neg$) has higher precedence than all other connectives.
- Question: Does $\neg p \wedge q$ mean (i) $\neg(p \wedge q)$ or (ii) $(\neg p) \wedge q$?
- Conjunction ($\wedge$) has next highest predence.
- Question: Does $p \wedge q \vee q$ mean (i) $(p \wedge q) \vee r$ or (ii) $p \wedge (q \vee r)$?
- Disjunction ($\vee$) has third highest precedence.
- Next highest is precedence is $\to$, and lowest precdence is $\leftrightarrow$

Operator Precedence Example

- Which is the correct interpretation of the formula

$$p \vee q \wedge r \leftrightarrow q \to \neg r$$

(A) $((p \vee (q \wedge r)) \leftrightarrow q) \to (\neg r)$

(B) $((p \vee q) \wedge r) \leftrightarrow q) \to (\neg r)$

(C) $(p \vee (q \wedge r)) \leftrightarrow (q \to (\neg r))$

(D) $(p \vee ((q \wedge r) \leftrightarrow q)) \to (\neg r)$

Summary

- Formulas in propositional logic are formed using propositional variables and boolean connectives
- Connectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$, conditional $\rightarrow$, biconditional $\leftrightarrow$
- Truth table shows truth value of formula under all possible assignments to variables