CS243: Discrete Structures

More Number Theory and Applications in Cryptography

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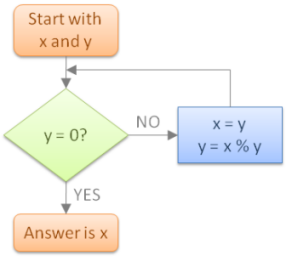
Announcements

- Fourth homework assignment handed out today
- Due October 18 (Thursday after fall break)
- Covers sequences, countability, number theory

Review of Last Lecture

- Congruence Modulo: $a \equiv b \pmod{m}$ iff $m|(a-b)$
- Alternatively, $a \equiv b \pmod{m}$ iff $a \bmod m = b \bmod m$
- $\gcd(a, b)$ is the largest integer d such that $d|a$ and $d|b$
- Theorem: Let $a = bq + r$. Then, $\gcd(a, b) = \gcd(b, r)$
- Euclid's algorithm is used to efficiently compute gcd of two numbers and is based on previous theorem.

Euclidian GCD Algorithm



- Find gcd of 72 and 20
- $12 = 72\%20$
- $8 = 20\%12$
- $4 = 12\%8$
- $0 = 8\%4$
- gcd is 4!

GCD as Linear Combination

- gcd(a, b) can be expressed as a linear combination of a and b
- Theorem: If a and b are positive integers, then there exist integers s and t such that:

$$\gcd(a, b) = s \cdot a + t \cdot b$$

- Furthermore, Euclidian algorithm gives us a way to compute these integers s and t

Example

- Express $\gcd(252, 198)$ as a linear combination of 252 and 198
- First apply Euclid's algorithm (write $a = bq + r$ at each step):
  1. $252 = 1 \cdot 198 + 54$
  2. $198 = 3 \cdot 54 + 36$
  3. $54 = 1 \cdot 36 + 18$
  4. $36 = 2 \cdot 18 + 0 \Rightarrow$ gcd is 18
- Now, using (3), write 18 as $54 - 1 \cdot 36$
- Using (2), write 18 as $54 - 1 \cdot (198 - 3 \cdot 54)$
- Using (1), we have $54 = 252 - 1 \cdot 198$, thus:

$$18 = (252 - 1 \cdot 198) - 1(198 - 3 \cdot (252 - 1 \cdot 198))$$

Example, cont.

$$18 = (252 - 1 \cdot 198) - 1(198 - 3 \cdot (252 - 1 \cdot 198))$$

- Now, let's simplify this:

$$18 = 252 - 1 \cdot 198 - 1 \cdot 198 + 3 \cdot 252 - 3 \cdot 198$$

- Now, collect all 252 and 198 terms together:

$$18 = 4 \cdot 252 - 5 \cdot 198$$

- Trace steps of Euclid's algorithm backwards to derive s, t:

$$\gcd(a, b) = s \cdot a + t \cdot b$$

- This is known as the extended Euclidian algorithm

A Useful Result

- Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.
- Proof: Since a, b are relatively prime $\gcd(a, b) = 1$
- By previous theorem, there exists s, t such that $1 = s \cdot a + t \cdot b$
- Multiply both sides by c: $c = csa + ctb$
- By earlier theorem, since $a|bc$, $a|ctb$
- Also, by earlier theorem, $a|csa$
- Therefore, $a|csa + ctb$, which implies $a|c$ since $c = csa + ctb$ □

Example

Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.

- Suppose $15 \mid 16 \cdot x$
- Here 15 and 16 are relatively prime
- Thus, previous theorem implies: $15|x$

Question

- Suppose $ca \equiv cb \pmod{m}$. Does this imply $a \equiv b \pmod{m}$?
- Counterexample: Consider $14 \equiv 8 \pmod{6}$
- Thus, $2 \cdot 7 \equiv 2 \cdot 4 \pmod{6}$
- But $7 \not\equiv 4 \pmod{6}$
- Therefore, this implication does not hold in the general case!
- However, if c and m are relatively prime, it does hold

Another Useful Result

- Theorem: If $ca \equiv cb \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$
- Proof: Since $ca \equiv cb \pmod{m}$, we have $m \mid ca - cb$
- Rewriting, we get: $m \mid c(a - b)$
- Since m, c are relatively prime, previous thm implies $m \mid a - b$
- By definition of congruence, $a \equiv b \pmod{m}$

Examples

- If $15x \equiv 15y \pmod{4}$, is $x \equiv y \pmod{4}$?
- If $8x \equiv 8y \pmod{4}$, is $x \equiv y \pmod{4}$?
- Counterexample: $8 \cdot 2 \equiv 8 \cdot 3 \pmod{4}$, but $2 \not\equiv 3 \pmod{4}$

Linear Congruences

- A congruence of the form $ax \equiv b \pmod{m}$ where a, b, m are integers and x a variable is called a linear congruence.
- Given such a linear congruence, often need to answer:
  1. Are there any solutions?
  2. What are the solutions?
- Observe: Determining if this congruence has a solution is the same as determining if the equality

$$ax - mk = b$$

has integer solutions.

Determining Existence of Solutions

- Theorem: The linear congruence $ax \equiv b \pmod{m}$ has solutions iff $\gcd(a, m)|b$.
- Proof involves two steps:
  1. If $ax \equiv b \pmod{m}$ has solutions, then $\gcd(a, m)|b$.
  2. If $\gcd(a, m)|b$, then $ax \equiv b \pmod{m}$ has solutions.
- First prove (1), then (2).

Proof, Part I

If $ax \equiv b \pmod{m}$ has solutions, then $\gcd(a, m)|b$.

- Suppose c is a solution, i.e., $ac \equiv b \pmod{m}$
- Then, $m|(ac - b)$
- Means there is a k such that $ac - b = mk$
- Rewrite as $b = ac - mk$
- $\gcd(a, m)|a$ and $\gcd(a, m)|m$; hence, $\gcd(a, m) \mid (ac - mk)$
- Since $b = ac - mk$, we have $\gcd(a, m)|b$ □

Proof, Part II

If $\gcd(a, m)|b$, then $ax \equiv b \pmod{m}$ has solutions.

- Let $d = \gcd(a, m)$ and suppose $d|b$
- Then, there is a k such that $b = dk$
- By earlier theorem, there exist s, t such that $d = s \cdot a + t \cdot m$
- Multiply both sides by k: $dk = a \cdot (sk) + m \cdot (tk)$
- Since $b = dk$, we have $b - a \cdot (sk) = m \cdot tk$
- Thus, $b \equiv a \cdot (sk) \pmod{m}$
- Hence, sk is a solution. □

Examples

- Does $5x \equiv 7 \pmod{15}$ have any solutions?
- Does $3x \equiv 4 \pmod{7}$ have any solutions?
- Note: This result generalizes to linear Diophantine equations
- Equality $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ has integer solutions iff

$$\gcd(a_1, a_2, \ldots, a_n)|b$$

- Previous result just an instance of this because $ax \equiv b \pmod{m}$ can be written as $ax - mk = b$

Examples

- Does $77x + 42y = 35$ have integer solutions?
- Does $6x + 9y + 12z = 7$ have integer solutions?

Finding Solutions

- Can determine existence of solutions, but how to find them?
- Theorem: Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod{m}$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

Example

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- What are the solutions to the linear congruence $3x \equiv 4 \pmod{7}$?
- First, need to find s, t such that $3s + 7t = gcd(3, 7)$
- Apply Euclidian algorithm: $7 = 2 \cdot 3 + 1$ and $3 = 3 \cdot 1 + 0$
- Hence $gcd(3, 7) = 1 = -2 \cdot 3 + 1 \cdot 7$. Thus, $s = -2$ and $t = 1$
- Solutions: $x = -2 \cdot 4 + 7u = -8 + 7u$ (e.g., $-8, -1, 6, 13, \ldots$)

Another Example

Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod{m}$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

- What are the solutions to the linear congruence $3x \equiv 1 \pmod{7}$?
- Already found s as -2 in previous example
- Solutions: $x = -2 + 7u$ (e.g, $-2, 5, 12, \ldots$)

Inverse Modulo m

- The inverse of a modulo m, written $\overline{a}$ has the property:

$$a\overline{a} \equiv 1 \pmod{m}$$

- Theorem: Inverse of a modulo m exists if and only if a and m are relatively prime.
- Proof: Inverse must satisfy $ax \equiv 1 \pmod{m}$
- By previous thm, this equation has a solution iff $\gcd(a, m)|1$
- Thus, $\gcd(a, m)$ must be 1 □
- Does 3 have an inverse modulo 7?

Example

- Find an inverse of 3 modulo 7.
- An inverse is any solution to $3x \equiv 1 \pmod{7}$
- Earlier, we already computed solutions for this equation as:

$$x = -2 + 7u$$

- Thus, -2 is an inverse of 3 modulo 7
- $5, 12, -9, \ldots$ are also inverses

Example 2

- Find inverse of 2 modulo 5.
- Need to solve the congruence $2x \equiv 1 \pmod{5}$
- What are s, t such that $2s + 5t = 1$?
- One solution: $\frac{1 \cdot 3}{1} = 3$
- Other solutions: $8, 13, 18, \ldots$

Solving Systems of Linear Congruences

- So far, learned how to solve single linear congruence
- In some cases, need to solve a system of linear congruences
- A famous theorem, called Chinese remainder theorem, tells us how to solve a system of linear congruences
- Chinese Remainder Theorem: Let m and n be relatively prime integers. Then, the system:

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$

has a solution. Furthermore, all solutions are congruent to $ant + bms$ modulo mn where $ms + nt = 1$.

Example

- Find all solutions to the following system:

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

- Find all solutions to first two, then combine with third
- Need to find s, t such that $s \cdot 3 + t \cdot 5 = 1$
- Using Euclid's algorithm: $5 = 1 \cdot 3 + 2$, $3 = 1 \cdot 2 + 1$, $2 = 2 \cdot 1 + 0$
- Applying backward substitution, we get: $1 = 2 \cdot 3 - 1 \cdot 5$
- Hence, $s = 2$, $t = -1$ and $ant + bms = -10 + 18 = 8$
- Thus, solution to first two congruences: $x \equiv 8 \pmod{15}$

Example, cont.

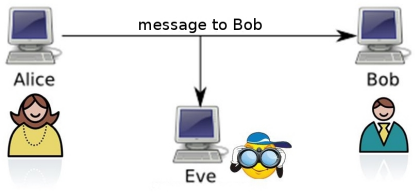
- Now, combine this with last congruence:

$$x \equiv 8 \pmod{15}$$
$$x \equiv 2 \pmod{7}$$

- Now, again find s, t such that $15s + 7t = 1$
- Applying Euclid, we have: $15 = 2 \cdot 7 + 1$, $7 = 7 \cdot 1 + 0$
- Hence, $1 = 1 \cdot 15 - 2 \cdot 7$, i.e. $s = 1, t = -2$
- $ant + bms = -112 + 30 = -82$
- Solution: $x \equiv -82 \pmod{105}$

Cryptography

- Cryptography is the study of techniques for secure transmission of information in the presence of adversaries



- How can Alice send secrete messages to Bob without Eve being able to read them?

Private vs. Public Crypto Systems

- Two different kinds of cryptography systems:
  1. Private (secret) key cryptography
  2. Public key cryptography
- In private key cryptography, sender and receiver agree on secret key that both use to encrypt/decrypt the message
- In public key crytography, a public key is used to encrypt the message, and private key is used to decrypt the message

Private Key Cryptography

- Private key crypto is classical method, used since antiquity
- Caesar's cipher is an example of private key cryptography
- Caesar's cipher is shift cipher where $f(p) = (p + k) \pmod{26}$
- Both receiver and sender need to know k to encrypt/decrypt
- Analogy: Alice wants to send Bob briefcase with secret message; they have a common key to lock/unlock briefcase
- Alice locks briefcase with shared key and Bob unlocks brief case with shared key
- Only works well when number of parties involved in communicated is small

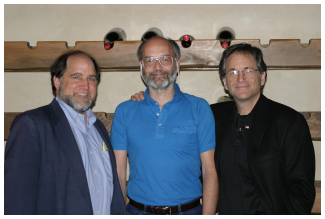
Public Key Cryptography

- Public key cryptography is the modern method, proposed by Diffie and Hellman in 70's
- Different keys are used to encrypt vs. decrypt message
- How can parties exchange information using different keys?
- Analogy: Alice puts message in briefcase, locks with her own key A, sends to Bob
- Bob gets locked briefcase, adds his lock B, sends back to Alice
- Alice gets double locked box, removes A, sends back to Bob
- Bob opens briefcase using his own key

Public Key Cryptography Overview

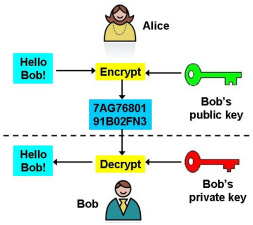
- This double lock example illustrates how parties can securely transmit information without exchanging secret keys
- Many modern crypo-systems work roughly this way
- Most commonly used public key system is RSA
- Great application of number theory and things we've learned

RSA History



- Named after its inventors Rivest, Shamir, and Adlemann, all researchers at MIT (1978)
- Actually, similar system invented earlier by British researcher Clifford Cocks, but classified – unknown until 90's

RSA Overview



- Bob has two keys: public and private
- Everyone knows Bob's public key, but only he knows his private key
- Alice encrypts message using Bob's public key
- Bob decrypts message using private key
- Public key can encrypt, but not decrypt
- Therefore, noone can read message accept Bob

High Level Math Behind RSA

- In the RSA system, private key consists of two very large prime numbers p, q
- Public key consists of a number n, which is the product of p, q and another number e
- e is a number relatively prime with $(p-1)(q-1)$ ($\phi(N)$, Euler's totient function)
- Encrypt messages using n, e, but to decrypt, must know p, q
- In theory, can extract p, q from n using prime factorization, but this is intractable for very large numbers
- **Security of RSA relies on inherent computational difficulty of prime factorization**

Encryption in RSA

- To send message to Bob, Alice first represents message as a sequence of numbers
- Call this number representing message M
- Alice then uses Bob's public key n, e to perform encryption as:

$$C = M^e \pmod{n}$$

- C is called the ciphertext

Encryption Example

- Encrypt message "STOP" using RSA with $n = 2537,\ e = 13$
- First convert each letter to a number in $[0, 25]$: $S = 18,\ T = 19,\ O = 14,\ P = 15$
- Group sequence into blocks of 4 digits:

$$M = 1819\ 1415$$

- Now encrypt each block as $C = M^{13} \pmod{2537}$
- For first block, $1819^{13} \pmod{2537} = 2081$; for second block $1415^{13} \pmod{2537} = 2182$
- Ciphertext: 2081 2182

RSA Decryption

- How do we decrypt cipher text using provate keys p, q?
- Decryption key d is the inverse of e modulo $(p-1)(q-1)$:

$$d \cdot e \equiv 1 \pmod{(p-1)(q-1)}$$

- As we saw earlier, d can be computed reasonably efficiently if we know $(p-1)(q-1)$
- However, since adversaries do not know p, q, they cannot compute d with reasonable computational effort!

RSA Decryption, cont.

- Using the Chinese remainder theorem and another theorem called Fermat's Little Theorem, it can be shown that:

$$(M^e)^d \equiv M \pmod{n}$$

- Since the ciphertext C is just M^e, $C^d \pmod{n}$ allows decrypting the message
- Since Bob can compute d using p, q, Bob can easily decrypt message, but no one else can!

Decryption Example

- Decrypt the cipher text 0981 0461 for the RSA cipher with $p = 43,\ q = 59$, and $e = 13$.
- First we need to compute d, the inverse of e modulo $(p-1)(q-1)$
- Here, $(p-1)(q-1) = 2436$; thus solve:

$$13x \equiv 1 \pmod{2436}$$

- To solve this, first compute s, t such that:

$$13s + 2436t = 1$$

- Apply extended Euclidian algorithm: $s = 937,\ t = -5$

Example, cont.

Decrypt 0981 0461 using $p = 43,\ q = 59, n = 2537$, and $e = 13$.

- To solve $13x \equiv 1 \pmod{2436}$, computed $s = 937,\ t = -5$
- Recall: Solution to this sytem is given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

- Here, $s = 937, b, d = 1, m = 2436$, thus solution: $\overline{e} = 937$
- $0981^{937} (\mathrm{mod}\ 2537) = 0704$; $0461^{937} (\mathrm{mod}\ 2537) = 1115$
- Thus, decrypted message is 0704 1115, or in English, "HELP"

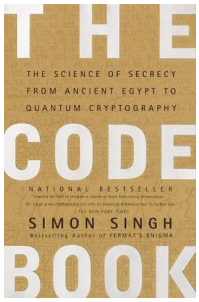
Security of RSA

- The encryption function used in RSA is a trapdoor function
- Trapdoor function is easy to compute in one direction, but very difficult in reverse direction without additional knowledge
- Encryption direction is easy because just requires exponentiation and mod
- Decryption without private key is very hard because requires prime factorization
- Therefore, security of RSA depends on difficulty of prime factorization

Security of RSA, cont.

- However, as computers get more powerful and factorization algorithms better, possible to factor larger and larger integers
- Therefore, over time, necessary to use larger and larger prime numbers to ensure secure communication
- For quantum computing, there are very efficient algorithms for computing prime factors (Shor's algorithm)
- If we could build quantum computers with sufficient "qubits", RSA would no longer be secure!
- However, today, RSA is considered secure if you use sufficiently large prime numbers (> 200 digits)

Book Recommendation



If you are interested in (history of) cryptography, read "The Code Book" by Simon Singh!