CS243: Discrete Structures

Strong Induction and Recursively Defined Structures

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Announcements

- Homework 4 is due today
- Homework 5 is out today
- Covers induction (last lecture, this lecture, and next lecture)
- Homework 5 due next Thursday Nov. 1
- 7 questions, all of them require proofs $\Rightarrow$ start early!

Review

- Induction is used to prove universally quantified properties about natural numbers and other countably infinite sets
- Consists of a base case and inductive step
- Base case: prove property about the least element(s)
- Inductive step: assume $P(k)$ and prove $P(k+1)$
- The assumption that $P(k)$ is true is called inductive hypothesis

Example (review)

- Prove the following statement by induction:

$$\forall n \in \mathbb{Z}^+ . \sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

- Base case: $n = 1$. In this case, $\sum_{i=1}^{1} i = 1$ and $\frac{(1)(1+1)}{2} = 1$; thus, the base case holds.
- Inductive step: By the inductive hypothesis, we assume $P(k)$:

$$\sum_{i=1}^{k} i = \frac{(k)(k+1)}{2}$$

- Now, we want to show $P(k+1)$:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Example (review), cont.

- First, observe:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

- By the inductive hypothesis, $\sum_{i=1}^{k} i = \frac{(k)(k+1)}{2}$; thus:

$$\sum_{i=1}^{k+1} i = \frac{(k)(k+1)}{2} + (k+1)$$

- Rewrite left hand side as:

$$\sum_{i=1}^{k+1} i = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

- Since we proved both base case and inductive step, property holds.

Plan for Today

- Strong induction
- Recursive definitions
- Proving properties of recursively defined functions

Strong Induction

- Strong induction is a proof technique that is a slight variation on matemathical (regular) induction
- Just like regular induction, have to prove base case and inductive step, but inductive step is slightly different
- Regular induction: assume $P(k)$ holds and prove $P(k+1)$
- Strong induction: assume $P(1), P(2), .., P(k)$; prove $P(k+1)$
- Regular induction and strong induction are equivalent, but strong induction can sometimes make proofs easier

Motivation for Strong Induction

- Prove that if n is an integer greater than 1, then it is either a prime or can be written as the product of primes.
- Let's first try to prove the property using regular induction.
- Base case (n=2): Since 2 is a prime number, $P(2)$ holds.
- Inductive step: Assume k is either a prime or the product of primes.
- But this doesn't really help us prove the property about $k+1$!
- Claim is proven much easier using strong induction!

Proof Using Strong Induction

Prove that if n is an integer greater than 1, then it is either a prime or can be written as the product of primes.

- Base case: same as before.
- Inductive step: Assume each of $2, 3, \ldots, k$ is either prime or product of primes.
- Now, we want to prove the same thing about $k+1$
- Two cases: $k+1$ is either (i) prime or (ii) composite
- If it is prime, property holds.

Proof, cont.

- If composite, $k+1$ can be written as pq where $2 \geq p, q \geq k$
  - Because a composite number has a divisor distinct from 1 and itself
- By the IH, p, q are either primes or product of primes.
- Thus, $k+1$ can also be written as product of primes
- Observe: Much easier to prove this property using strong induction!

A Word about Base Cases

- In all examples so far, we had only one base case
  - i.e., only proved the base case for one integer
- In some inductive proofs, there may be multiple base cases
  - i.e., prove base case for the first m numbers
- Perfectly fine to have inductive proofs with multiple base cases
- In such proofs, inductive step only needs to consider numbers greater than m

Example

- Prove that every integer $n \geq 12$ can be written as $n = 4a + 5b$ for some non-negative integers a, b.
- Proof by strong induction on n and consider 4 base cases
- Base case 1 (n=12):
- Base case 2 (n=13):
- Base case 3 (n=14):
- Base case 4 (n=15):

Example, cont.

Prove that every integer $n \geq 12$ can be written as $n = 4a + 5b$ for some non-negative integers a, b.

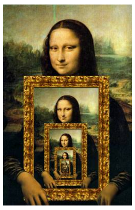
- Inductive hypothesis: Suppose every $12 \leq i \leq k$ can be written as $i = 4a + 5b$.
- Inductive step: We want to show $k + 1$ can also be written this way for $k + 1 \geq 16$
- Observe: $k + 1 = (k - 3) + 4$
- By IH, $k - 3 = 4a + 5b$ for some a, b because $k - 3 \geq 12$
- But then, $k + 1$ can be written as $4(a + 1) + 5b$
- Would this proof work if we only showed base case for $n = 12$?

Recursive Definitions

- In this class, we saw how to define functions and sequences:
  - $f(n) = 2n$
  - $a_n = 3n + 1$
- These are examples of direct (non-recursive) definitions
- But in some cases, it is easier to define functions/sets in terms of themselves rather than directly

Recursive Definitions

- Definitions of structures that refer to themselves are called recursive definitions
- Picture below is "defined" recursively because each picture containts a smaller version of itself



Recursive Definitions in Math

- Recursive definitions come up a lot in discrete math
- Consider the factorial function: $n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$
- This is a direct definition, but easier to define factorial recursively:

$$\begin{aligned} 1! &= 1 \\ n! &= (n-1)! \cdot n \end{aligned}$$

- Definition is recursive because we use ! when defining !

General Structure of Recursive Definitions

- Recursive definitions consist of base case and recursive step
- Base case defines function for least element in the domain
- Recursive step shows how to compute $f(k + 1)$ assuming $f(k)$ can be computed
- For factorial, base case is $1! = 1$
- For factorial, recursive step is $n! = (n - 1)! \cdot n$
- Recursive definitions are similar to proofs by induction
- In fact, recursive definitions sometimes called inductive definitions

Recursively Defined Function Example

- Here is another recursive definition:

$$\begin{aligned} f(0) &= 3 \\ f(n+1) &= 2f(n) + 3 \quad (n \geq 1) \end{aligned}$$

- What is $f(1)$?
- What is $f(2)$?
- What is $f(3)$?

Recursively Defined Sequences

- Just like functions, sequences can also be defined recursively
- For example, consider the following sequence:

$$1, 3, 9, 27, 81, \ldots$$

- What is a recursive definition of this sequence?
- Base case:
- Recursive step:

Recursive Definition Examples

- Consider $f(n) = 2n + 1$ where n is non-negative integer
- What's a recursive definition for f?
- Consider the sequence $1, 4, 9, 16, \ldots$
- What is a recursive definition for this sequence?
- Recursive definition of function defined as $f(n) = \sum_{i=1}^{n} i$?

Recursive Definitions of Important Functions

- Some important functions/sequences defined recursively
- Factorial function:

$$\begin{aligned} f(1) &= 1 \\ f(n) &= n \cdot f(n-1) \quad (n \geq 2) \end{aligned}$$

- Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \quad (n \geq 2) \end{aligned}$$

- Just like there can be multiple bases cases in inductive proofs, there can be multiple base cases in recursive definitions

Inductive Proofs for Recursively Defined Structures

- Recursive definitions and inductive proofs are very similar
- Therefore, it's natural to use induction to prove properties about recursively defined functions and sequences
- In these proofs, base case of induction shows property holds for base case of recursive definition
- Similarly, the inductive step shows the property holds for the recursive part of the definition

Example 1

- Consider the function defined recursively as follows:

$$\begin{aligned} f(0) &= 1 \\ f(n) &= f(n-1) + 3 \end{aligned}$$

- Prove that $f(n) = 3n + 1$
- We'll prove this by regular mathematical induction
- Base case:

Example 1, cont.

$$\begin{aligned} f(0) &= 1 \\ f(n) &= f(n-1) + 3 \end{aligned}$$

- Inductive step: We need to show $f(k+1) = 3(k+1) + 1$ assuming $f(k) = 3k + 1$
- Using the recursive case of definition, $f(k+1) = f(k) + 3$
- From IH, $f(k) = 3k + 1$
- Thus, $f(k+1) = 3k + 1 + 3 = 3(k+1) + 1$ □

Example 2

- Let f_n denote the n'th element of the Fibonacci sequence
- Prove: For $n \geq 3$, $f_n > \alpha^{n-2}$ where $\alpha = \frac{1+\sqrt{5}}{2}$
- Proof is by strong induction on n with two base cases
- Base case 1 (n=3): $f_3 = 2$, and $\alpha < 2$, thus $f_3 > \alpha$
- Base case 2 (n=4): $f_4 = 3$ and $\alpha^2 = \frac{(3+\sqrt{5})}{2} < 3$

Example 2, cont.

Prove: For $n \geq 3$, $f_n > \alpha^{n-2}$ where $\alpha = \frac{1+\sqrt{5}}{2}$

- Inductive step: We need to show $f_{k+1} > \alpha^{k-1}$ for $k+1 > 4$
- Using IH, we can assume $f_i > \alpha^{i-2}$ for $3 \leq i \leq k$
- First, rewrite α^{k-1} as $\alpha^2 \alpha^{k-3}$
- α^2 is equal to $1+\alpha$ because:

$$\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{\sqrt{5}+3}{2} = \alpha + 1$$

- Thus, $\alpha^{k-1} = (\alpha+1)(\alpha^{k-3}) = \alpha^{k-2}\alpha^{k-3}$

Example, cont.

- $\alpha^{k-1} = \alpha^{k-2}\alpha^{k-3}$
- By recursive definition, we know $f_{k+1} = f_k + f_{k-1}$
- Furthermore, by inductive hypothesis:

$$f_k > \alpha^{k-2} \qquad f_{k-1} > \alpha^{k-3}$$

- Therefore, $f_{k+1} > \alpha^{k-2}\alpha^{k-3} = \alpha^{k-1}$

□

Recursively Defined Sets and Structures

- We saw how to define functions and sequences recursively
- We can also define sets and other data structures recursively
- Example: Consider the set S defined as:

$$3 \in S$$
$$\text{If } x \in S \text{ and } y \in S, \text{ then } x+y \in S$$

- What is the set S defined as above?

More Examples

- Give a recursive definition of the set E of all even integers:
  - Base case:
  - Recursive step:
- Give a recursive definition of $\mathbb{N}$, the set of all natural numbers:
  - Base case:
  - Recursive step:

Strings and Alphabets

- Recursive definitions play important role in study of strings
- Strings are defined over an alphabet Σ
  - Example: $\Sigma_1 = \{a, b\}$
  - Example: $\Sigma_2 = \{0\}$
- Examples of strings over Σ_1: $a, b, aa, ab, ba, bb, \ldots$
- Set of all strings formed from Σ forms language called Σ^*
  - $\Sigma_2^* = \{\epsilon, 0, 00, 000, \ldots\}$

Recursive Definition of Strings

- The language Σ^* has natural recursive definition:
  - Base case: $\epsilon \in \Sigma^*$ (empty string)
  - Recursive step: If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$
- Since ϵ is the empty string, $\epsilon s = s$
- Consider the alphabet $\Sigma = \{0, 1\}$
- How is the string "1" formed according to this definition?
- How is "10" formed?

Recursive Definitions of String Operations

- Many operations on strings can be defined recursively.
- Consider function $l(w)$ which yields length of string w
- Example: Give recursive definition of $l(w)$
  - Base case:
  - Recursive step:

Another Example

- The reverse of a string s is s written backwards.
- Example: Reverse of "abc" is "bca"
- Give a recursive definition of the $\text{reverse}(s)$ operation
  - Base case:
  - Recursive step:

Palindromes

- A palindrome is a string that reads the same forwards and backwards
- Examples: "mom", "dad", "abba", "Madam I'm Adam", ...
- Give a recursive definition of the set P of all palindromes over the alphabet $\Sigma = \{a, b\}$
- Base cases:
- Recursive step: