# CS243: Discrete Structures

## Combinatorics

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## Introduction

- Consider a set of objects and a property of interest
- Often, we want to know how many of these objects have the desired property
- Example: How many possible passwords are there if a password must contain 6-8 characters?
- These kinds of problems arise frequently in computer science
- Combinatorics (counting) deals with these questions



## Basic Counting Rules

- Counting problems can be hard $\Rightarrow$ useful to decompose
- Two basic very useful decomposition rules:
  1. Product rule: useful when task decomposes into a sequence of independent tasks
  2. Sum rule: decomposes task into a set of alternatives



## Product Rule

- Suppose a task $A$ can be decomposed into a sequence of two independent tasks $B$ and $C$
- $n_1$ ways of doing $B$
- $n_2$ ways of doing $C$
- Product rule: Then, there are $n_1 n_2$ ways of doing $A$



## Example 1

- New company with 12 offices and 2 employees Kate and Jack
- How many ways to assign different offices to Kate and Jack?
- Decomposition: First assign office to Kate, then to Jack
- How many ways to assign office to Kate?
- How many ways to assign office to Jack?
- How many ways to assign offices to Kate and Jack?



## Example 2

- Chairs in auditorium labeled with a letter (A-Z) and an integer $\in [1, 100]$.
- Examples: D32, G4, Z99
- Under this labeling scheme, what is the maximum number of chairs in auditorium if we want every chair to be labeled?
- Observe: Max # of labeled chairs = # of different labelings
- Same as "How many different labelings for a chair?"
- Decomposition: First assign letter, then integer to chair

## Example 2, cont.

Chairs in auditorium labeled with a letter (A-Z) and an integer $\in [1, 100]$. How many different labels?

- How many ways to assign a letter?
- How many ways to assign digit?
- How many ways to assign label?



## Extended Product Rule

- We formulated product rule for decomposition into 2 tasks
- But generalizes to decomposition into any $k$ tasks
- Suppose task $A$ decomposes to sequence of tasks $A_1, \ldots A_k$
- If there are $n_1$ ways of doing $A_1$, ..., $n_k$ ways of doing $A_k$, then there are $n_1 \cdot n_2 \ldots \cdot n_k$ ways of doing $A$



## Example 3

- A bitstring is a string where each character is either $0$ or $1$
- How many different bit strings of length $7$ are there?
- Decomposition: First assign bit to first character, then to second, ..., then to seventh
- How many ways to assign bit to each character?
- How many different bitstrings?
- By extended product rule: $2 \cdot 2 \ldots 2$ (7 times)
- This is $2^7 = 128$



## Example 4

- Each license plate consists of $3$ letters followed by $3$ digits [0-9]
- How many different license plates are there?
- Decomposition:
- How many ways to assign each letter?
- How many ways to assign each digit?
- How many different labels for each license plate?



## Counting One-to-One Functions

- How many one-to-one functions are there from a set with $3$ elements to a set with $5$ elements?
- Recall: In a one-to-one function, different elements in domain cannot be assigned to same element in codomain
- Decomposition: Assign first element in domain, then second element, then third element
- How many ways to assign first element?
- How many ways to assign second element?
- How many ways to assign third element?
- How many different functions?



## Sum Rule

- Two basic very useful decomposition rules:
  1. Product rule ✓
  2. Sum rule
- Suppose a task $A$ can be done either in way $B$ or in way $C$
- Suppose there are $n_1$ ways to do $B$, and $n_2$ ways to do $C$
- Sum rule: There are $n_1 + n_2$ ways to do $A$.

## Example 1

- Suppose either a CS faculty or CS student must be chosen as representative for a committee
- There are 14 faculty, and 50 majors
- How many ways are there to choose the representative?
- Note: Just like the product rule, the sum rule can be extended to more than two tasks
- Extended sum rule: If task can be done in one of $n_1$ or $n_2$, or ..., $n_k$ ways, then there are $n_1 + n_2 + \ldots + n_k$ ways of doing it



## Example 2

- A student can choose a senior project from one of three lists
- First list contains 23 projects; second list has 15 projects, and third has 19 projects
- Also, no project appears on more than one list
- How many different projects can student choose?
- What if some of the projects appeared on both lists?
- Caveat: For sum rule to apply, the possibilities must be mutually exclusive



## More Complex Counting Problems

- Problems so far required either only product or only sum rule
- But more complex problems require a combination of both!
- Example: In a programming language, a variable name is a string of one or two characters.
- A character is either a letter [a-z] or a digit [0,9].
- First character in the string must be a letter.
- How many possible variable names are there?



## Example, cont.

- First, decompose using sum rule.
- Variable name is either one character or two characters
- Compute number $n_1$ of one character names
- Compute number $n_2$ of two character names
- By sum rule, there is $n_1 + n_2$ names
- First compute $n_1$: How many 1 character names are there?



## Example, cont.

- Now, compute $n_2$: How many 2 character names are there?
- For this, decompose using product rule: assign first character first, then second character
- How many ways to assign first character?
- How many ways to assign second character?
- How many 2 character names?
- By sum rule, total number of variable names = $n_1 + n_2 = 26 + 936 = 962$



## Another Example

- A password must be six to seven characters long
- A character is upper case letter or digit
- Each password must contain at least one digit
- How many possible passwords?
- First decompose using sum rule: either six or seven characters
- Thus, number of possible passwords is $n_6 + n_7$ where $n_i$ is number of passwords with $i$ characters

## Example, cont.

- Computing $n_6$: How many six-character passwords are there containing at least one digit?
- For this, compute the total number of six-character passwords; then subtract number of passwords without any digits
- Total # of 6-char passwords:
- # of 6-char passwords without any digits:
- Thus, $n_6 = 36^6 - 26^6$



## Example, cont.

- Computing $n_7$: How many seven-character passwords are there containing at least one digit?
- $n_7$ = Total - # without any digits
- Total # of 7-char passwords =
- Those without any digits:
- $n_7 = 36^7 - 26^7$
- Total number of passwords = $n_6 + n_7 = 72,200,220,480$



## Example 3

- How many bitstrings are there of length $6$ that do not have two consecutive $1$'s?
- Let $F(n)$ denote the number of bitstrings of length $n$ that do not have two consecutive $1$'s
- We'll first derive a recursive equation to characterize $F(n)$
- By the sum rule, $F(n)$ is the sum of:
  1. # of $n$-bit strings starting with $1$ not containing $11$
  2. # of $n$-bit strings starting with $0$ not containing $11$



## Example 3, cont.

- How many n-bit strings are there starting with $1$ and not containing $11$?
- Since we don't want two consecutive $1$'s, second bit must be $0$
- But the rest can be anything as long as it doesn't contain two consecutive $1$'s
- Thus, the rest = number of bitstrings of length $n-2$ not containing two consecutive $1$'s
- But this is precisely $F(n-2)$!
- Thus, by product rule, there are $1 \cdot 1 \cdot F(n-2) = F(n-2)$ n-bit strings starting with $1$ and not containing $11$



## Example 3, cont.

- How many $n$-bit strings are there starting with $0$ and not containing $11$?
- First bit must be $0$ but rest can be anything as long as it doesn't contain $11$
- How many ways of forming "the rest"?
- # of bitstrings of length $n-1$ not containing $11$:
- By product rule, number of $n$-bit strings starting with 0, not containing $11$ is $1 \cdot F(n-1) = F(n-1)$



## Example 3, cont.

- Recall: $F(n)$ is sum of:
  1. # of $n$-bit strings starting with $1$ not containing $11$
  2. # of $n$-bit strings starting with $0$ not containing $11$
- We determined that (1) is $F(n-2)$
- We determined that (2) is $F(n-1)$
- Therefore, $F(n) = F(n-1) + F(n-2)$
- This is a recursive definition, but what are the base cases?

## Example 3, cont.

- What is $F(1)$?
- What is $F(2)$?
- Original question asks for $F(6)$
- Using base cases and recursive definition:
  - What is $F(3)$?
  - What is $F(4)$?
  - What is $F(5)$?
  - What is $F(6)$?



## Recall: Sum Rule

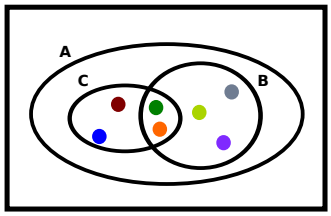
- Recall: Sum rule only applies if a task is as disjunction of two mutually exclusive tasks
- What do we do if the tasks aren't mutually exclusive?
- Example: You can choose from set $A$ or set $B$, but they have some elements in common
- In such cases, the sum rule is not applicable because common elements counted twice
- Fortunately, there is a generalization of the sum rule called the inclusion-exclusion principle



## The Inclusion-Exclusion Principle

- Suppose a set $A$ can be written as union of sets $B$ and $C$
- Inclusion-Exclusion Principle:

$$|A| = |B| + |C| - |B \cap C|$$






## Inclusion-Exclusion Principle Example

- How many bit strings of length $8$ either start with $1$ or end with two bits $00$?
- Let $B$ be the set of bitstrings that start with $1$
- Let $C$ be the set of bitstrings that end with $00$
- We want $|B \cup C|$
- By the inclusion-exclusion principle, $|B \cup C| = |B| + |C| - |B \cap C|$
- Thus, compute $|B|, |C|$ and $|B \cap C|$



## Example, cont.

- First compute $|B|$, bitstrings of length $8$ that start with $1$
- What is $|B|$?
- Now compute $|C|$, bitstrings of length $8$ ending in $00$
- What is $|C|$?
- Now compute $|B \cap C|$, those starting with $1$ and ending in $00$
- What is $|B \cap C|$?
- Number of 8-bit srings that start with $1$ and end in $00$:

$$128 + 64 - 32 = 160$$



## Another Example

- A company receives $350$ applications for job positions
- $220$ of applicants are CS majors, $147$ of applicants are business majors, and $51$ are double CS and business majors
- How many are neither CS nor business majors?
- # of non-CS/non-business applicants = Total applicants - # of CS or business majors
- What is $|B \cup CS|$?
- The remaining applicants are neither business nor CS:

$$350 - 316 = 34$$

## The Pigeonhole Principle

- Suppose there is a flock of 36 pigeons and a set of 35 pigeonholes
- Each pigeon wants to sit in its own hole
- But since there are less holes than there are pigeons, one pigeon is left without a hole.

- The Pigeonhole Principle: If $n+1$ or more objects are placed into $n$ boxes, then at least one box contains 2 or more objects



## Examples

- Consider an event with 367 people. Is it possible no pair of people have the same birthday?
- Consider function $f$ from a set with $k+1$ or more elements to a set with $k$ elements. Is it possible $f$ is one-to-one?
- Consider $n$ married couples. How many of the $2n$ people must be selected to guarantee there is at least one married couple?



## Generalized Pigeonhole Principle

- If $n$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil n/k \rceil$ objects
- Proof: (by contradiction) Suppose every box contains less than $\lceil n/k \rceil$ objects
- Then, there must be less than $k \cdot (\lceil n/k \rceil - 1)$ objects
- From earlier, we know $\lceil \frac{n}{k} \rceil < \frac{n}{k} + 1$
- Adding one and multiply both sides by $k$: $k \cdot (\lceil \frac{n}{k} \rceil - 1) < n$
- Thus, this would imply there are less than $n$ objects, a contradiction



## Examples

- If there are 30 students in a class, at least how many must be born in the same month?
- What is the minimum # of students required to ensure at least 6 students receive the same grade (A, B, C, D, F)?
- Want min $n$ such that $\lceil \frac{n}{5} \rceil = 6 \Rightarrow$
- What is the min # of cards that must be chosen to guarantee three have same suit? (suits: $\clubsuit, \heartsuit, \spadesuit, \diamondsuit$)
- Want min $n$ such that $\lceil \frac{n}{4} \rceil = 3 \Rightarrow$



## More Examples

- Suppose there are 25 million phones in a state
- Each phone number has seven digits, where first digit is $[2-9]$ and other digits are $[0-9]$
- What is the minimum number of area codes necessary to guarantee each phone has a unique number?
- First, compute # of seven-digit phone numbers:
- Let $k$ be the number of area codes
- Want to find minimum $k$ such that $\left\lceil \frac{25 \times 10^6}{k} \right\rceil \leq 8 \times 10^6 \Rightarrow k = 4$