CS243: Discrete Structures

Propositional Logic II

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Announcements

- First homework assignment out today!
- Due in one week, i.e., before lecture next Tuesday 09/11
- Weilin's Tuesday office hours are 9-10 AM, not 10-11 AM

Review

- Propositional logic is simplest kind of logic
- Building blocks are propositions, i.e., statements that are true or false
- Formulas in propositional logic are formed using propositional variables and boolean connectives
- Connectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$, conditional $\rightarrow$, biconditional $\leftrightarrow$
- Truth table shows truth value of formula under all possible assignments to variables

Operator Precedence

- Given a formula $p \wedge q \vee r$, do we parse this as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$?
- Without settling on a convention, formulas without explicit paranthesization are ambiguous.
- To avoid ambiguity, we will specify precedence for logical connectives.
- Operator precedence is a convention that tells us how to parse formulas if they are not explicitly parenthesized.

Operator Precedence, cont.

- Negation ($\neg$) has higher precedence than all other connectives.
- Question: Does $\neg p \wedge q$ mean (i) $\neg(p \wedge q)$ or (ii) $(\neg p) \wedge q$?
- Conjunction ($\wedge$) has next highest predence.
- Question: Does $p \wedge q \vee r$ mean (i) $(p \wedge q) \vee r$ or (ii) $p \wedge (q \vee r)$?
- Disjunction ($\vee$) has third highest precedence.
- Next highest is precedence is $\rightarrow$, and lowest precdence is $\leftrightarrow$

Operator Precedence Example

- Which is the correct interpretation of the formula

$$p \vee q \wedge r \leftrightarrow q \rightarrow \neg r$$

(A) $((p \vee (q \wedge r)) \leftrightarrow q) \rightarrow (\neg r)$

(B) $((p \vee q) \wedge r) \leftrightarrow q) \rightarrow (\neg r)$

(C) $(p \vee (q \wedge r)) \leftrightarrow (q \rightarrow (\neg r))$

(D) $(p \vee ((q \wedge r) \leftrightarrow q)) \rightarrow (\neg r)$

Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- Example: $\neg p$ evaluates to true under under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for every assignment, e.g., $p \vee \neg p$
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., $p \wedge \neg p$
- Such formulas are called unsatisfiable formulas or contradictions

Interpretations

- Concepts of validity, satisfiability are very important in logic!
- To make them precise, we'll define interpretation of formula
- An interpretation I for a formula F is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \cdots\}$$

- In general, for formula with n propositional variables, there are 2^n interpretations
- Each interpretation corresponds to one row in the truth table

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

Formula F + Interpretation I = Truth value

- We write $I \models F$ if F evaluates to true under I
- Similarly, $I \not\models F$ if F evaluates to false under I.
- Theorem: $I \models F$ if and only if $I \not\models \neg F$

Examples

- Consider the formula $F: \ p \wedge q \rightarrow \neg p \vee \neg q$
- Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{false}]$
- What does F evaluate to under I_1?
- Thus, $I_1 \models F$
- Let I_2 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$
- What does F evaluate to under I_2?
- Thus, $I_2 \not\models F$

Another Example

- Let F_1 and F_2 be two propositional formulas
- Suppose F_1 evaluates to true under interpretation I
- What does $F_2 \wedge \neg F_1$ evaluate to under I?

Satisfiability, Validity

- F is satisfiable iff there exists interpretation I s.t. $I \models F$
- F is valid iff for all interpretations I, $I \models F$
- F is unsatisfiable iff for all interpretations I, $I \not\models F$
- F is contingent if it is satisfiable, but not valid.
- Valid formulas also called tautologies
- Unsatisfiable formulas called contradictions

True/False Questions

Are the following statements true or false?

- If a formula is valid, then it is also satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable.
- If F_1 and F_2 are satisfiable, then $F_1 \wedge F_2$ is also satisfiable.
- A formula can be both contingent and unsatisfiable

Duality Between Validity and Unsatisfiability

F is valid if and only if $\neg F$ is unsatisfiable

- Proof:
  - By definition, F is valid iff for all interpretations I, $I \models F$
  - By theorem, $I \models F$ iff $I \not\models \neg F$
  - Thus, F is valid iff for all interpretations I, $I \not\models \neg F$
  - But if for all interpretations I, $I \not\models \neg F$, then $\neg F$ is unsat
  - Thus, F valid iff $\neg F$ unsat

Proving Validity

- Question: How can we prove that a propositional formula is a tautology?
- Exercise: Which formulas are tautologies? Prove your answer.
  1. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
  2. $(p \wedge q) \vee \neg p$

Proving Satisfiability, Unsatisfiability, Contingency

- Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:
  - Satisfiable: There exists a row where formula evaluates to true
  - Unsatisfiable: In all rows, formula evaluates to false
  - Contingent: Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercises

1. Prove $\neg(p \wedge q) \wedge \neg(\neg p \vee \neg q)$ is unsatisfiable
2. Prove $(p \rightarrow q) \rightarrow (q \rightarrow p)$ is a contingency

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$F_1 \Rightarrow F_2$ iff $F_1 \rightarrow F_2$ is valid

- Caveat: $F_1 \Rightarrow F_2$ is not a propositional logic formula; $\Rightarrow$ is not part of PL syntax!
- Instead, $F_1 \Rightarrow F_2$ is a semantic judgment, like satisfiability!

Syntax vs. Semantics

- Syntax: What you are allowed to write
  - $\land, \rightarrow$ are part of PL syntax, but $\star, \Rightarrow$ are not!
  - $p_1 \land p_2$ is a syntactically valid PL formula, $p_1 \star p_2$ is not!
- Semantics: Concerns meaning of what is written
  - Validity, satisfiability semantic notions b/c they concern meaning of the formula
  - Semantics gives meaning to syntax
- Difference between syntax vs. semantics crucial in CS
  - Comes up in logic, programming languages, theory of computation, ...

Checking Implication

- Question: How can we check if $F_1 \Rightarrow F_2$?
- Exercise: Does $p \lor q$ imply p? Prove your answer!

Equivalence

- Consider two propositional formulas F_1 and F_2.
- Sometimes F_1 and F_2 always have same truth value for every interpretation, e.g., $p \lor p$ and $p \land p$
- Such formulas F_1 and F_2 called equivalent, written $F_1 \equiv F_2$ or $F_1 \Leftrightarrow F_2$
- More precisely, formulas F_1 and F_2 are equivalent iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

$$F_1 \Leftrightarrow F_2 \text{ iff } F_1 \leftrightarrow F_2 \text{ is valid}$$

- $\equiv, \Leftrightarrow$ not part of PL syntax; they are semantic judgments!

Checking Equivalence

- Question: How can we prove $F_1 \equiv F_2$?
- Exercise: Prove $p \rightarrow q$ and $\neg p \lor q$ are equivalent

Important Equivalences

- Some important equivalences are useful to know!
- Law of double negation: $\neg\neg p \equiv p$
- Identity Laws: $p \land T \equiv p$ $\quad p \lor F \equiv p$
- Domination Laws: $p \lor T \equiv T$ $\quad p \land F \equiv F$
- Idempotent Laws: $p \lor p \equiv p$ $\quad p \land p \equiv p$
- Negation Laws: $p \land \neg p \equiv F$ $\quad p \lor \neg p \equiv T$

Commutativity and Distibutivity Laws

- Commutative Laws: $p \lor q \equiv q \lor p$ $\quad p \land q \equiv q \land p$
- Distributivity Law #1: $(p \lor (q \land r)) \equiv (p \lor q) \land (p \lor r)$
- Distributivity Law #2: $(p \land (q \lor r)) \equiv (p \land q) \lor (p \land r)$
- Associativity Laws: $p \lor (q \lor r) \equiv (p \lor q) \lor r$
  $p \land (q \land r) \equiv (p \land q) \land r$

De Morgan's Laws

- Let cs243 be the proposition "John took CS243" and cs303 be the proposition "John took CS303"
- In simple English what does $\neg(cs243 \wedge cs303)$ mean?
- DeMorgan's law expresses exactly this equivalence!
- De Morgan's Law #1: $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
- De Morgan's Law #2: $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
- When you "push" negations in, $\wedge$ becomes $\vee$ and vice versa

Using Equivalences

- We saw one way to prove two formulas are equivalent: use truth table
- Another way: use known equivalences to rewrite one formula as the other
- Examples: Prove following formulas are equivalent using known equivalences.
  1. $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$
  2. $\neg(p \rightarrow q)$ and $p \wedge \neg q$

Formalizing English Arguments in Logic

- We can use logic to prove correctness of English arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let f be the proposition "Joe drives fast", and t be the proposition "Joe gets a ticket"
- How do we encode this argument as a logical formula?

Example, cont

"If Joe drives fast, he gets a speeding ticket. Joe did not get a ticket. Therefore, he did not drive fast.": $((f \rightarrow t) \wedge \neg t) \rightarrow \neg f$

- How can we prove this argument is valid?
- Can do this in two ways:
  1. Use truth table to show formula is tautology
  2. Use known equivalences to rewrite formula to true
- Let's use equivalences

Another Example

- Can also use to logic to prove an argument is not valid.
- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Let's use logic to prove George's argument doesn't hold water.
- Let u = "Jill is carrying an umbrella", and r = "It is raining"
- How do we encode this argument in logic?

Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining.": $((u \rightarrow r) \wedge \neg u) \rightarrow \neg r$

- How can we prove George's argument is invalid?

Summary

- A formula is valid if it is true for all interpretations.
- A formula is satisfiable if it is true for at least one interpretation.
- A formula is unsatisfiable if it is false for all interpretations.
- A formula is contingent if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas F_1 and F_2 are equivalent, written $F_1 \equiv F_2$, if $F_1 \leftrightarrow F_2$ is valid